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# Essays on Dynamic Portfolio Management

*Chien-Hui Liao*

A thesis submitted to  
The University of Warwick  
for the degree of  
Doctor of Philosophy

May 2003

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Over the last three decades, there has been an increasing interest in the problem of the investor's optimal consumption and portfolio rules. Despite the substantial amount of related literature, there remain many areas for further investigation. The thesis, therefore, addresses a number of important issues relating to the theory and practice of dynamic portfolio strategies.

The thesis consists of five essays. The first two essays, Chapters 3 and 4, are concerned with efficient dynamic asset allocation programs under alternative market assumptions. Chapter 3 studies a situation where the simple time-invariant portfolio strategies are efficient and provides a complete characterisation of the strategies using the efficiency arguments. The popularised constant proportion portfolio insurance (CPPI) is embedded as a special case. Chapter 4 relaxes the assumption of a constant interest rate to allow the interest rate to follow a one-factor stochastic process. The factor risk premium is then determined in a way that is consistent with the underlying equilibrium. These results are then applied to solve explicitly for an investor's optimal portfolio choice problem under the special case of a Vasicek short rate model and alternative utility functions.

The third essay, Chapter 5, relaxes the assumption of a constant equity risk premium to allow the risk premium to vary through time. The evolution of the market risk premium in a representative agent equilibrium (consistent with the Black-Scholes option pricing) is investigated using a unified approach. The presence of dividends and intermediate consumption proves to be the key element that enables us to obtain a stationary economy with decreasing relative risk aversion, a theoretical result that has not been established in the existing literature.

The last two essays, Chapters 6 and 7, are concerned with issues of portfolio efficiency and performance measurement. Chapter 6 uses the result from Chapter 5 that, without dividends and intermediate consumption, the market risk premium must satisfy the Burgers' equation, and applies Dybvig's payoff distribution pricing model to measure the inefficiency costs incurred when this condition is violated. The numerical results show that the degree of inefficiency is not very significant.

at least for the cases which we postulate. but the findings also reassure negative result predicted from the model.

Finally, Chapter 7 proposes a new utility based performance measure that can be applied in the ex-post evaluation of dynamic portfolio strategies. We construct a contingent claim estimation approach to estimate the nearest efficient strategy from a single realisation and then quantify the opportunity cost resulting from the departure of the observed strategy from the nearest efficient one. The numerical examples show that the technique is remarkably robust.

**Keywords:**

Dynamic Asset Allocation, Equilibrium, Time-Varing Risk Premium, Portfolio Efficiency, Performance Measurement



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I would like to dedicate this thesis to my parents. Without their unconditional love and continuous support, I would not have been able to complete this work. In particular, my father passed away when I was about to conclude the thesis. I pray that his soul rests in peace and this humble thesis can bring him some comfort. I would also like to thank my husband for his understanding and full support - intellectually and domestically.

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I declare that this thesis is submitted to the University of Warwick for the degree of Doctor of Philosophy in 2003. Except where acknowledged, the material contained in this thesis is my own work and that it has neither been previously published nor submitted elsewhere for the purpose of obtaining an academic degree.

*Chien-Hui Liao*

# Chapter 1

## Introduction

### 1.1 Introduction and Scope of Research

The thesis addresses a number of important theoretical issues in the field of investment management. Given the extent of this field, our focus in this work is mainly on the issues relating to the theory and practice of dynamic asset allocation.

As is widely recognised, the very notion of uncertainty plays a fundamental role in the process of financial decision making. Moreover, time and preferences are also central to the problem. The interrelations between these three make the problem complex and therefore, pose a great challenge in the search of optimal rules for efficiently allocating financial resources across time.

The dynamic and complex relationships between uncertainty, time, and preferences also indicate that a portfolio is more likely to be managed in a dynamic fashion. A dynamic asset allocation program is basically a trading rule that defines the way how the portfolio manager should adjust the holdings across different

asset classes on a frequent basis. However, only those which optimise at least one rational investor's expected utility can be called efficient strategies.

Efficient portfolio strategies need to be characterised and identified if possible under alternative market assumptions. The properties of an efficient strategy can help identify whether or not a particular strategy is efficient. If it turns out to be inefficient, then the question would be: how inefficient is it? can we measure the magnitude of the inefficiency?

In a complete market, a setting which will be used universally in the thesis, a simple maximisation of (non-state dependent) expected utility of final wealth gives the well known equation:

$$U'(W_s) = \lambda \left( \frac{q_s}{p_s} \right). \quad (1.1)$$

This means that at each state  $s$ , the investor's marginal utility of wealth  $W$  is proportional to the state-price density (the ratio of the state price  $q$  to the state probability  $p$ ). Dybvig [36, 37] exploits this relation and proposes that an efficient wealth must be monotonically decreasing in the state-price density.

Another well-known property of an efficient portfolio strategy is path-independence. This property was first proposed by Cox and Leland [29]. It means that all trajectories with the same terminal value must have the same probability.

Equipped with the above insights, our starting point is to reinvestigate time-invariant strategies since these are one of the simplest and popular rules. This class of strategies has a close relationship to the asymptotic portfolio theory established in 1970's. However, the literature is somewhat confused about the exact working



of this type of strategies. Therefore, Chapter 3 aims to provide a formal analysis and clarification of the issue.

The setting in Chapter 3 is rather simple in the sense that the state-price density is assumed to be a function of the market index only. Therefore, more complicated situations ought to be considered. Among a variety of possible ways of relaxing assumption, stochastic interest rates seem more interesting and important in the context of intertemporal analysis. For this reason, Chapter 4 introduces stochastic interest rates and solves explicitly the dynamic asset allocation problem under the Vasicek's one-factor short rate model. A significant part of the analysis is that the factor risk premium is determined in such a way that it is consistent with the underlying equilibrium. This has introduced what would be necessary for equilibrium asset price processes. We continue developing this concept further in Chapter 5.

As is known, the risk premium is extremely important for portfolio management. In Chapters 3 and 4, it is assumed that the equity risk premium is constant. However, there is some evidence that the expected risk premium varies through time in a way which displays mean reversion. The evolution of the risk premium over time has been studied previously by He and Leland [57], Hodges and Carverhill [61] and Hodges and Selby [63]. These authors laid out an important theoretical foundation for this direction of research. They discovered that in a Black and Scholes economy without dividends and intermediate consumption (an economy characterised by a representative agent who maximises his/her expected utility of terminal wealth), the risk premium must satisfy a non-linear partial differential

equation called Burgers' equation in a representative agent equilibrium. However, a decreasing risk premium is not possible in a stationary economy.

Therefore, in Chapter 5, we first give a new exposition which provides a unified approach. We then introduce dividends and intermediate consumption and address the question: with dividends, can we now obtain a stationary economy with decreasing relative risk aversion? The answer to this question is an encouraging yes!

A direct implication from Chapter 5's analysis is that, if the risk premium does not evolve following the necessary equation, then the model implies the market portfolio will become inefficient. Therefore, in Chapter 6, we examine the simple one-consumption case and ask the question: if the risk premium has a plausible but non-equilibrium structure (i.e. it does not satisfy the Burgers' equation), how inefficient is the market portfolio? We directly make use of Dybvig's monotonicity result and calculate the exact magnitude of the inefficiency cost. We show that the degree of inefficiency is not very significant, but the findings also reassure negative result predicted from the model.

As the second application of Dybvig's work, Chapter 7 turns to the issue of performance measurement. Performance measurement is of considerable importance in its own right. The increasing popularity of hedge funds further calls for more sophisticated performance evaluation procedures as they generally possess very different characteristics from conventional mutual funds.

The problem is less difficult if the portfolio strategy is known. However, it is



almost impossible to obtain this information in practice. Moreover, although the concept of our model is close to Sharpe's [98] style analysis, data limitations make style analysis difficult or even impossible. An alternative approach is needed in order to make the model operational. As a new contribution, we apply Dybvig's concept of portfolio efficiency and propose a new performance measure which can be applied in the ex-post evaluation.

## 1.2 Organisation of Thesis

The thesis is presented in eight chapters. In an attempt to improve the legibility of the thesis, the main body is divided into three parts:

Part 1 [Chapter 3, Chapter 4]. Optimal dynamic asset allocation rules.

Part 2 [Chapter 5]. Time-varying market risk premium.

Part 3 [Chapter 6, Chapter 7]. Portfolio efficiency and performance measurement.

In Chapter 2, a review of the literature relating to the thesis work in the following three areas is provided: (1) dynamic asset allocation; (2) equilibrium asset price processes; and (3) portfolio efficiency and performance measurement.

Chapters 3 and 4 concern efficient dynamic asset allocation rules. In Chapter 3, we study a situation where time-invariant portfolio strategies are efficient and provide a complete characterisation of the strategies. We arrive at the important subclass strategy - the constant proportion portfolio insurance (CPPI) policy - and discuss how it relates to other strategies such as the stop-loss rule and the perpetual American call options.

Chapter 4 explores how the equilibrium framework can be used to solve dynamic asset allocation problems. We consider an economy in which interest rates are stochastic. The factor risk premium is determined in a way that is consistent with the underlying equilibrium. We then apply the martingale approach to solve explicitly for the optimal portfolio choice under the assumption of a Vasicek [103] short rate model.

In Chapter 5, we investigate the theoretical constraints on the time variation in the risk premia of the market portfolio in a Black and Scholes economy. We characterise the equilibrium conditions as nonlinear partial differential equations and solve for closed-form stationary solutions. The intermediate consumption is included to provide a richer behaviour – diminishing in the market level – that can not be obtained in a single consumption economy.

Chapters 6 and 7 address issues concerning portfolio performance measurement. In Chapter 6, we postulate the risk premium functions motivated by the results from Chapter 5. We apply Dybvig's performance measure and employ the Monte Carlo simulations to evaluate the efficiency of the market portfolio.

In Chapter 7, we propose a new utility based performance measure and explore the extent to which it can be applied in the ex-post evaluation. We construct a contingent claim estimation approach to approximate the opportunity cost implied by inefficient dynamic strategies. The technique is implemented on some simple but popular strategies such as the stop-loss rule and lock-in strategy through Monte Carlo simulations. The results demonstrate how the pathwise inefficiency cost will distribute across each eventuality. The average pathwise cost is also compared



with the Dybvig's global cost.

Chapter 8 concludes the thesis and indicates some directions for future research.

# Chapter 2

## Literature Review

In this section, we provide a survey of the financial theory relating to dynamic portfolio management. The amount of related literature is considerable. Furthermore, the related work is scattered in the literature with different emphases. Therefore, we divide the survey into the following three areas: (1) dynamic asset allocation; (2) equilibrium asset price processes; and (3) portfolio efficiency and performance measurement.

### 2.1 Dynamic Asset Allocation

Dynamic asset allocation (DAA) can be defined, according to Trippi and Harrif [102], as ‘a class of investment strategies that shifts the content of portfolios between two or more asset classes in response either to changes in the value of the portfolio and/or external economic states, on a more or less continual basis’. As stated in Hodges [60], the motivation of conducting DAA is two-fold: (1) to tailor the distribution of fund return at some future date so that it can be an entirely

different shape from that of the market index; (2) to exploit predictable market regularities.

According to the above definition and motivation, we may interpret a DAA program as some kind of device that aims to ‘maximise’ its investors’ profits by skillfully adjusting the asset mix over time. Therefore, it is closely related to the portfolio choice problem, which has been studied extensively over the last three and a half decades or so.

Earlier studies on the problem of optimal portfolio strategies for long-lived investors such as Latane and Tuttle [76], Mossin [89], Hakansson [52], Samuelson [94] are normally conducted in either a static environment or a discrete-time setting. Returns are usually assumed to be i.i.d.

Merton [83, 84] pioneered the powerful continuous-time technique and analysed the portfolio choice problem in an intertemporal fashion. He shows in [84] that if preferences and future investment opportunity sets are state-independent, then intertemporal portfolio optimisation can be treated as if the investor had a single-period utility function (see also Fama [45]). However, if the investor faces a changing opportunity set, his or her optimal portfolio will behave very differently.

Merton [85] further shows that when the investment opportunity set is stochastic, the investor will want to hedge against the unfavourable changes in the state variables. As a result, the optimal portfolio consists of an instantaneously mean-variance efficient portfolio (or the myopic portfolio) and a hedge portfolio. His assumption of an interest rate following a geometric Brownian motion is perhaps



one of the earliest papers to assume stochastic interest rates. Based on the work of Merton, the optimal dynamic portfolio choice problem has become an active area of later research.

From late 1970's, a particular type of dynamic investment strategy, usually referred to as 'portfolio insurance' emerged and became a popular subject both for academicians and practitioners. In general, portfolio insurance involves frequent trading. It typically buys stocks when their prices rise and sell stocks when their prices fall. The result of purchasing portfolio insurance is a convex payoff function. The earliest important research articles were by Brennan [11], Brennan and Schwartz [13], Brennan and Solanki [16] and Leland [77]. Most of these authors, except Leland, studied what kind of insurance contract should an investor buy given his preferences (in terms of utility functions) and beliefs (in terms of return generating processes). The question addressed by Leland instead was more general, 'who should buy portfolio insurance?' He concluded that: (1) investors who have average expectations, but with risk tolerance increasing with wealth more rapidly than average, will wish to buy portfolio insurance; (2) investors who have average risk tolerance, but with expectations of returns more optimistic than average, will wish to buy portfolio insurance.

The popularity of portfolio insurance started to climb to the peak from the mid 1980's. In 1988, Brennan and Schwartz [14] claimed to have characterised a general model for time-invariant portfolio insurance strategies. Few years later, Black and Perold [9] gave a theory of constant proportion portfolio insurance (CPPI). The two papers provide valuable insights into the theoretical justifications of the

portfolio insurance program. Although portfolio insurance is a simple type of investment strategy, there is still a need for some clarification and bring the literature together. Chapter 3 of this thesis therefore addresses this issue by providing a general characterisation of time-invariant portfolio strategies.

Since the crash of October 1987, it has been suggested that certain types of dynamic trading strategies, in particular, portfolio insurance and program trading, tend to increase market volatility. Therefore, these dynamic strategies have been accused of contributing to the market crash by increasing volatility and having a general distablising effect on the market. (See for example, the Brady report [81].) As a consequence, a number of researchers attempted to study the impact of portfolio insurance on market equilibrium. (For example, Basak [3], Donaldson and Uhlig [35] and Grossman and Zhou [51].)

In recent years, attention has been paid in relaxing assumptions of Merton's original work on the optimal portfolio choice. For example, Kim and Omberg [73] and Wachter [104] solve a two-asset problem by assuming that the interest rate is constant and the equity premium follows an O-U process. Omberg [90] and Sørensen [100] instead assume that the equity premium is constant and the interest rate follows a Vasicek [103] one-factor process. Liu [80] solves a cash-bond allocation problem under the assumption of an affine term structure. Brennan and Xia [17] solves a three-asset problem assuming a two-factor interest rate model. Chapter 4 of this thesis provides a further contribution to this part of the literature by using an equilibrium approach and solving the three-asset portfolio choice problem of a non-representative agent in an economy with stochastic interest rates.



Motivated by empirical work which indicates apparent asset return predictability, Brennan, Schwartz and Lagnado [15] propose a model of economy where there are multiple state variables and solve numerically for a three-asset allocation problem. Their model gives the result that shows a great deal of volatility in the optimal holdings.

By assuming that the investor might be uncertain about the asset returns, Barberis [2], Brennan [12], Kandel and Stambaugh [72], Merton [84] and Xia [105] apply the Bayesian analysis to propose some kind of learning model that attempts to explain the return predictability.

A recent paper by Cvitanić, Goukasian and Zapatero [32] utilises the dual approach of Cox and Huang [25] to devise a numerical method for efficiently solving the intertemporal optimisation problem. Campbell, *et al.* [19] and Campbell and Viceira [21] assume an Epstein-Zin utility [43, 44] and propose to approximate the utility function and solve the problem numerically. The former paper considers stock market mean reversion and solve for the optimal equity allocation of a long-lived investor. The latter one instead solves the demand function of the long term bonds.

Finally, in the incomplete market setting, Brennan and Xia [18] are concerned with borrowing and short sales constraints on dynamic trading. Stochastic volatility is considered in Chacko and Viceira [23].

## 2.2 Equilibrium Asset Price Processes

Asset pricing models derived in a general equilibrium framework such as the classic papers by Cox, Ingersoll and Ross [27], Harrison and Krep [56], Huang [64] and Merton [85] establish relationship between the asset price processes and the economic fundamental variables. Therefore, they have the advantage over the arbitrage-free asset pricing models in that the pricing systems are internally consistent with the underlying equilibrium.

In a rather different setting, Cox and Leland [29] first showed that in an economy that satisfies the assumptions for Black-Scholes option pricing (i.e. a constant risk-free rate and constant volatility of the equity market), a dynamic portfolio strategy must be path independent to be efficient. The efficiency here refers to the first-order stochastic dominance, rather than the conventional mean-variance efficiency. This important property was later applied by others and could be easily shown through the following simple maximisation.

Assume an investor aims to maximise a non-state dependent expected utility of final wealth. The first order condition gives

$$U'(W_s) = \lambda \left( \frac{q_s}{p_s} \right). \quad (2.1)$$

This suggests that the ratio of the risk-neutral probability  $q$  to the objective probability  $p$  should be the same for the same aggregate wealth level (without loss of generality, we can assume the interest rate is zero). In other words, all the trajectories that end up with the same value must have the same probabilities. This *path-independence* result has had important implications on the analysis of



portfolio efficiency.

For example, in the analysis of market portfolio, Bick [4] applied this property and proposed a systematic approach to examining whether a given price process is consistent with equilibrium. Specifically, he showed that the Black-Scholes model is supported by a representative agent who has a CRRA utility. The approach is then generalised in Bick [5] to models with more general diffusion price processes.

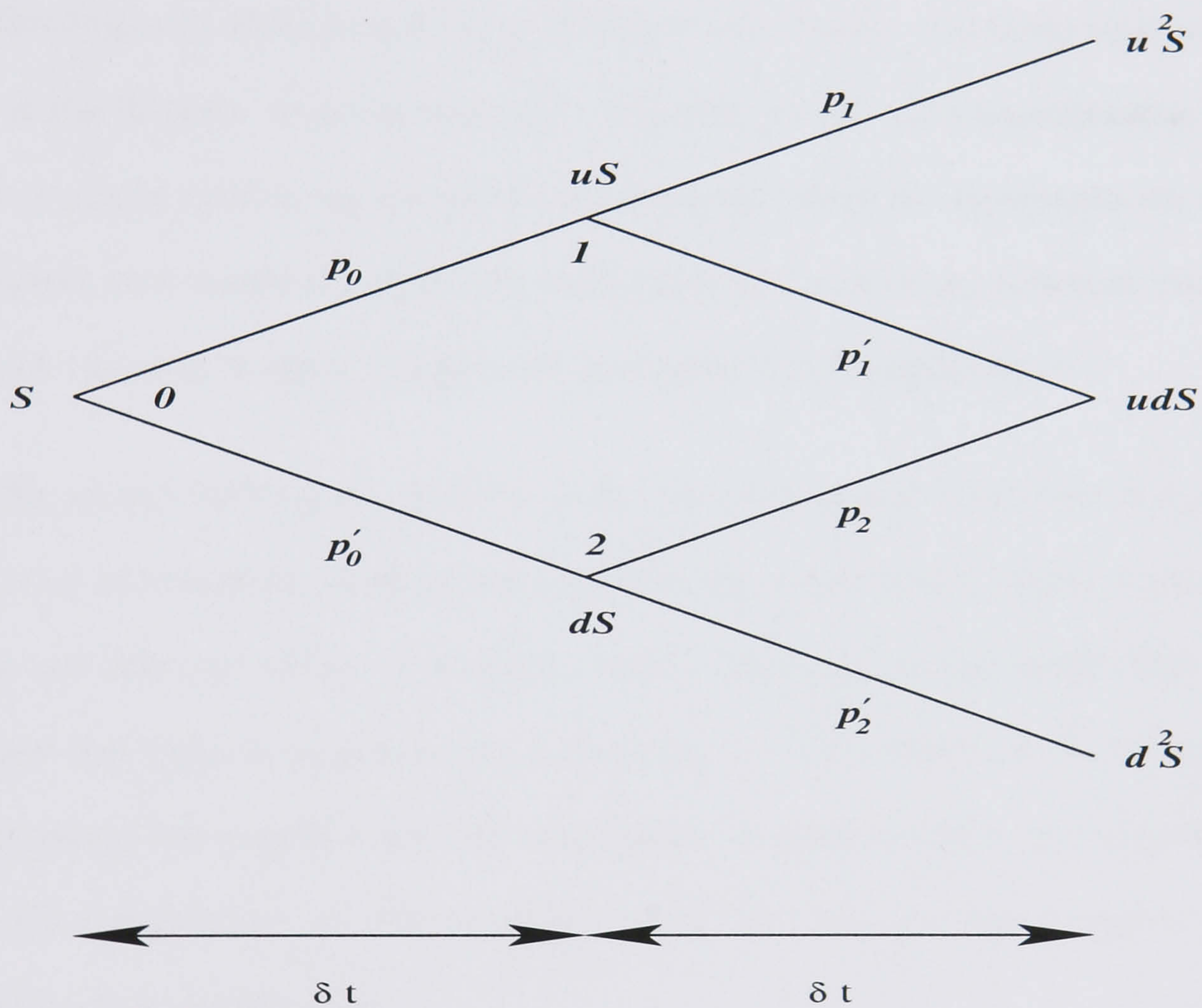


Figure 2.1: Binomial tree.

Hodges and Carverhill [61] also applied the path-independence result to investigate the evolution of the market risk premium in a simple representative agent economy. They begin with a simple discrete-time setting where the asset price follows a binomial process (see Figure 2.1). From the path-independence result



(which implies that  $p_0 p'_1 = p'_0 p_2$ , where  $p$ 's are objective probabilities). they then arrive at a difference equation which must be satisfied by the market risk premium. By taking the limit, they obtain an interesting continuous-time result. It turns out that in the limit, the difference equation becomes a partial differential equation called the Burgers equation. This equation is also proposed by another important work of He and Leland [57].

Subsequently, Hodges and Selby [63] provide a direct continuous-time derivation of the Burgers' equation and solve explicitly for the time-homogeneous case. They conclude that in the Black-Scholes economy where the representative agent maximises over terminal wealth, the only viable and stationary solutions are that the risk premium is either constant or increasing in the market level.

The above results were obtained under the assumptions that there is no dividend and intermediate consumption and that the representative agent maximises his or her expected utility of terminal wealth. Chapter 5 of this thesis, therefore, extends the analysis to encompass a situation in which there are dividends and intermediate consumption and the representative agent's utility is a function of both the consumption and the terminal wealth. The result of this extension turns out to be very encouraging.

## 2.3 Portfolio Efficiency and Performance Measurement

In the previous section, we have reviewed the path-independence result of Cox and Leland [29], which is a necessary condition for an investment strategy to be efficient in the sense that it will not throw away the investor's money. Dybvig [36] applied this result, together with the monotonicity property, to derive (in a complete market setting) a pricing model called 'payoff distribution pricing model' (PDPM).<sup>1</sup> Essentially, the PDPM gives the cheapest price of purchasing a consumption bundle when agents only concern with the distribution of terminal consumption.

Hence, the magnitude of the inefficiency cost of a consumption bundle (or the payoff generated by some strategy) can be defined as the difference between the asset price (or the investment required to replicate the strategy) and the distributional price (the cheapest price to 'buy' the same distribution). Denote by  $\xi$  the state-price density (or state price per unit probability) and by  $W$  the terminal wealth generated by some portfolio strategy from an initial investment  $w_0$ . By the (positive) linear pricing rule, the asset price is  $P_A(W; \xi) = \mathbb{E}[\xi W] = w_0$ . Since the distributional price is defined as the cheapest price for buying the same distribution  $F_W$ , the pricing function can be expressed as  $P_D(F_W; F_\xi) \equiv \min \left\{ P_A(\tilde{W}; \xi) \mid \tilde{W} \sim F_W \right\}$ . It turns out that this minimum cost is achieved when the terminal consumption is reversely related to the state-price density. That is,

$$P_D(F_W; F_\xi) = \int_{\gamma=0}^1 F_\xi^{-1}(\gamma) F_W^{-1}(1 - \gamma) d\gamma. \quad (2.2)$$

---

<sup>1</sup>The underlying assumptions can be found in Dybvig [36].



This leads to a new measure of portfolio performance (assuming no forecasting ability). The neutral performance occurs when  $P_D = P_A$ . When  $P_D < P_A$ , it indicates that the strategy is inefficient and the payoff  $W$  is stochastically dominated by trading in the benchmark market.

Dybvig [37] then further applied the PDPM to examine the degree of inefficiency incurred by following some popular dynamic investment strategies such as stop-loss, lock-in, random timing and repeated portfolio insurance. It is demonstrated that these strategies are significantly inefficient.

Prior to Dybvig's proposal of PDPM, conventional risk adjusted performance measures such as those proposed by Jensen [67], Sharpe [96, 97, 99] and Treynor [101] are based on the Capital Asset Pricing Model (CAPM) and, therefore, are based on the assumption that the market portfolio is mean-variance efficient. This will be an inappropriate assumption in an intertemporal setting. The problems associated with the CAPM based measures have been discussed in Dybvig and Ross [38, 42, 41] and Grinblatt and Titman [50]. The problem that a nonlinear type of payoff cannot be spanned in the CAPM framework is also documented in Merton [87].

In decomposing performance in different sources, papers by Merton [87] and Henriksson and Merton [58] propose an econometric procedure to evaluate the managers market timing ability. They demonstrate that the service values provided by market timing can be distinguished from selectivity. However, Admati *et al.* [1] and Jagannathan and Korajczyk [65] argue that this kind of decomposition is theoretically difficult and can result in erroneous conclusions. For example,

writing covered calls will arrive at the conclusion that the manager has inferior market time ability and superior selectivity if we use H-M procedure. This wrong assignment when the manager has not contributed any ability at all is primarily due to the nonnormal returns distribution.

In a recent paper by Leland [78], it is also demonstrated that a non-linear payoff generated from some option-like strategy will be wrongly evaluated in the mean-variance framework. For example, a concave payoff will be overrated and a convex payoff will be underrated by the CAPM. This distributional problem is partially resolved by Dybvig's [36] PDPM. (Notice that the PDPM appeared some 20 years after the traditional reward-to-variability measures were proposed.) The analysis in Dybvig [37] is to some extent in the spirit of Sharpe's [98] *style* analysis. Sharpe proposes an asset class factor model to help determine how effectively individual fund managers have performed their functions and the extent to which value has been added through active management.

A number of papers, such as Glosten and Jagannathan [48] and Hodges [59], have proposed contingent claim estimation procedures for evaluating the performance of a portfolio. The technique of Hodges [59] also enables the analyst to recover the portfolio manager's objective. In this respect, it is related to the work on recoverability of preferences by Dybvig [38] and Dybvig and Rogers [39].

Among other important work on portfolio performance measurement, Chen and Knez [24] provide a characterisation of the set of admissible performance measures which seem quite general. Stochastic discount factor (SDF) based performance measures are also often used. It usually involves estimating the SDF as

in Hansen and Jagannathan [55]. A recent paper by Söderlind [95] demonstrates that the SDF measure is essentially a performance measure derived from a factor portfolio model. The unconditional SDF measure corresponds to a Jensen's alpha for a fund, and the conditional SDF measures to a vector of Jensen's alphas for managed portfolios of funds.



# Chapter 3

## On the Time-Invariant Portfolio Strategies

This chapter and the next explore the issues relating to dynamic asset allocation (DAA) programs. In this chapter, we begin with a simple situation where the assumptions of the Black-Scholes model hold. Our focus is on time-invariant portfolio strategies. The conditions for efficiency enable us to characterise efficient time-invariant portfolio strategies and analyse the behaviour of the wealth functions. While the results are derived under strong assumptions, the model displays a richer behaviour than previously found in the related literature. The more complex problem of dynamic asset allocation under stochastic interest rates will be dealt with in Chapter 4.

### 3.1 Introduction

Asset allocation is an important part of investment management process. It is concerned with dividing the investor's funds among two or more asset class port-

folios. As described in the previous chapter, dynamic asset allocation (DAA) is concerned with adjusting the asset mix on an almost continual basis. Over the years, numerous financial economists and practitioners have proposed (and implemented) a variety of dynamic investment strategies. Some of these strategies are simple, others are not. Moreover, not all of these strategies are efficient. Therefore, as a starting point, it may be useful to investigate the simplest kinds of dynamic portfolio strategies that are efficient in terms of expected utility maximisation.

In principle, if we stay within the expected utility paradigm, then the ‘optimal’ portfolio strategy for an investor can be found by solving the intertemporal consumption and portfolio rules that maximise expected utility. The seminal paper by Merton [84] pioneered this technique. In a Black-Scholes world, it is found that the optimal strategy for an investor with a power utility of terminal wealth is to invest a constant proportion of wealth in the risky asset.

The idea that the constant policy could also be suitable for long horizon investment goes back to the asymptotic portfolio theory established since 1970’s.<sup>1</sup> The asymptotic portfolio theory is also called *portfolio turnpike theory* since it advocates that the coefficient of relative risk aversion (or the *portfolio turnpike*) can be found to converge to the corresponding power utility when the horizon is distant.

In practice, investors usually prefer skewed payoffs (or returns) which will not fall below some specified (nonnegative) level at some specified date. In that case,

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<sup>1</sup>See Cox and Huang [26], Dybvig, Rogers and Back [40], Hakansson [53, 54], Jin [68] and Ross [93], for a development of the theory.



the standard constant policy might be inappropriate since the rate of convergence of the investor's actual optimal strategy to the turnpike could be fairly slow. (Dybvig, Rogers and Back [40] show that it could be as long as 50 years!) Therefore, the non-zero minimum requirement should be taken into account in determining the portfolio strategy if desired. A variation of the constant policy which can overcome the slow convergence problem is the constant proportion portfolio insurance (CPPI) strategy popularised by Black and Jones [7, 8] and later examined by Black and Perold [9]. It invests a constant multiple of the cushion in risky assets up to the borrowing limit, where the cushion is the difference between wealth and a specified floor. It is also shown by Dybvig, Rogers and Back [40] that the CPPI rule is optimal for an investor with a *translated* CRRA utility function.

The above constant policies have the obvious advantage of being time-invariant. Time-invariant portfolio strategies are of interest because they are easy to implement (as alternative to complex option replication strategies) and more importantly they are useful for strategies that have indefinite horizons (relevant for some managed funds). Brennan and Schwartz [14] provide a general characterisation of time-invariant portfolio strategies and analyse the behaviour of the wealth function. Since their characterisation is deduced from the asymptotic portfolio theory, the nonzero requirement can not always be met. Moreover, it is not clear under what conditions the proposed strategy will be efficient.

In this chapter, we stay with the Black-Scholes assumptions and argue that the simplest kinds of dynamic portfolio strategies are those which are time-invariant. (The constant policy is obviously one of them.) Our approach is related to but

different from those of Brennan and Schwartz [14] and Black and Perold [9] in that the model is built on the efficiency conditions, i.e. non-negativity, monotonicity and path-independence. It turns out that our characterisation of efficient time-invariant portfolio strategies addresses and corrects the problem of Brennan and Schwartz [14] in the sense that the proposed strategies can accommodate non-zero minimum requirement and at the same time efficient. The CPPI strategy is, not surprisingly, a subclass of time-invariant strategies. However, our characterisation also displays a richer set of behaviour than previously found in the literature in that the shape of the wealth function can be concave, linear or convex. As in Leland [77], our model shows that investors who are more risk averse (that is, those who prefer a concave payoff function) than the representative agent will sell portfolio insurance to investors who are less risk averse.

The remainder of the chapter is organised as follows. In Section 3.2, we provide a characterisation of time-invariant portfolio strategies in a Black-Scholes economy without dividend and intermediate consumption. By the efficiency arguments, we then derive explicitly in Section 3.3 the wealth functions of time-invariant strategies that are consistent with expected utility maximisation. Section 3.4 visualises the results by applying realistic parameters to the relevant functions and evaluating the model numerically. Section 3.5 discusses how the model can be modified when a constant floor is preferred and how the constant strategy from our model can be related to investment in the perpetual American call options. Section 3.6 concludes the chapter.



## 3.2 Time-Invariant Portfolio Strategies

In a perfect market setting, consider an investor who has a positive initial wealth  $W_0$  and wishes to have a guaranteed minimum level of wealth  $K$  at the horizon date  $T$ . Assume that there are two assets in the economy available for trading: one non-dividend-paying risky asset (the stock index) and one riskless asset (the bond or a bank account). The value of the risky asset  $S$  is assumed to follow a geometric Brownian motion

$$\frac{dS}{S} = \mu dt + \sigma dz \quad (3.1)$$

where  $\mu$  and  $\sigma$  are constant and  $z$  is a standard Brownian motion under the objective probability measure  $\mathbb{P}$ . The price of the bond  $B$  increases at the constant rate  $r$  over time:

$$\frac{dB}{B} = r dt. \quad (3.2)$$

Over the investment horizon, the investor's wealth will accumulate according to some *self-financing* portfolio strategy. This means that there are no intermediate capital injections into or withdrawals from the 'fund'. Denote by  $W(t)$  the wealth level, or the value of the fund at time  $t$ . The number of shares of the risky asset held at time  $t$  is  $\Delta(t)$ . The risk exposure, i.e., the amount of money invested in the risky asset, is  $A(t)$ . With the minimum guarantee in mind, we can construct the strategy in the following way. The wealth value  $W(t)$  can be decomposed into two parts: a hypothetical level of  $F(t)$  which aims to achieve the future minimum requirement  $K$ , and a contingent claim  $C(t)$  which generates a terminal payoff dependent upon the value of the reference portfolio,  $S(T)$  at  $T$ . For convenience,

$F(t)$  will sometimes be referred to as the *floor* and  $C(t)$  the *cushion*.

From the above construction, the wealth function can be expressed as  $W(S, t) = F(t) + C(S, t)$  and the boundary conditions are that  $W(S, 0) = W_0$ ,  $F(T) = K$  and  $W(0, T) = K$ . We first note that the floor can not be of any arbitrary value. In fact, if the investor invests 100% of her initial wealth in the riskless asset, then she will get  $W_0 e^{rT}$  at  $T$  for certain. Therefore, the index-linked performance provided by the cushion  $C$  must be obtained at the expense of a lower minimum,  $K$ . This imposes a constraint on the floor, i.e.  $K \leq W_0 e^{rT}$ . We will see later that the floor  $F(t)$  must grow at the riskless interest rate.

Moreover, note that the wealth is specified as a function of  $t$  and  $S$  only. This is of significant importance because it directly implies that the wealth function is *path-independent* and Cox and Leland [29, 30] have shown that path-independence is a necessary condition for a strategy to be efficient in a Black-Scholes world. This specification also implies that the wealth function  $W(S, t)$  must satisfy the usual partial differential equation of Black and Scholes. The following lemma states this result.

**Lemma 3.1** *For a self-financing continuous trading investment strategy without borrowing restrictions, the value of the wealth  $W$  is a function of  $S$  and  $t$  only, i.e.,  $W(t) \equiv W(S, t)$  and satisfies the Black-Scholes second-order partial differential equation:*

$$W_t + rSW_S + \frac{1}{2}\sigma^2 S^2 W_{SS} - rW = 0. \quad (3.3)$$

*An amount of  $A(S, t) = \Delta(S, t) \cdot S$  is invested in the stock where  $\Delta(S, t) =$*



$W_S(S, t)$  is the number of shares of the stock held at  $t$ .

*Proof.* By construction, the wealth level is a function of  $S$  and  $t$  only, i.e.,  $W(t) = W(S, t)$ . Therefore, by *Ito's Lemma*, we have that

$$dW = W_S dS + \left( W_t + \frac{1}{2} \sigma^2 S^2 W_{SS} \right) dt. \quad (3.4)$$

Denote by  $\Delta(t)$  the number of shares of the stock held at time  $t$ . The increment of wealth is given by

$$dW = \Delta dS + (W - \Delta S) r dt. \quad (3.5)$$

Equating the coefficients of  $dS$  and  $dt$  in (3.4) and (3.5) yields  $\Delta(S, t) = W_S(S, t)$  and the partial differential equation (3.3).  $\square$

A natural interpretation of a time-invariant portfolio strategy is that the strategy itself is independent of time. Conventionally, a portfolio strategy is often expressed in terms of the proportions of wealth allocated across the asset classes. In our setting, however, we need to modify this in order to take account of the floor. The formal definition is given as follows.

**Definition 3.1** *Let  $x$  denote the proportion of wealth above the floor invested in the risky asset at time  $t$  according to some portfolio strategy. A portfolio strategy is called time-invariant if  $x$  is at most a function of the current risky asset value  $S$ . That is,  $x \equiv x(S)$ .*

Recall that  $C(S, t) = W(S, t) - F(t)$  and  $A(S, t) = \Delta(S, t)S = W_S(S, t)$ . Since  $W_S(S, t) = C_S(S, t)S$ , following the above definition of a time-invariant portfolio

strategy,  $x$  must satisfy

$$x(S, t) = \frac{A(S, t)}{W(S, t) - F(t)} = \frac{C_S(S, t)S}{C(S, t)} \equiv x(S). \quad (3.6)$$

A stationary solution for  $C(S, t)$  to satisfy (3.6) is that the cushion  $C(S, t)$  is multiplicatively separated into the function of time and the function of the current risky asset value, i.e.,

$$C(S, t) = g(t)h(S). \quad (3.7)$$

Thus, the wealth function is of the form:

$$W(S, t) = F(t) + g(t)h(S), \quad (3.8)$$

subject to the boundary conditions  $W(S, 0) = W_0$  and  $W(0, T) = F(T) = K$ .

According to our definition, by using (3.8) to solve the PDE (3.3), we can obtain a complete characterisation of the wealth function of time-invariant strategies. The result is formally stated in the following theorem.

**Theorem 3.1** *The wealth function under a generalised (borrowing unconstrained) time-invariant portfolio strategy is of the form:*

$$W(S, t) = F(t) + e^{\gamma t} (\beta_1 S^{\alpha_1} + \beta_2 S^{\alpha_2}), \quad (3.9)$$

where  $\gamma = (1 - \alpha_i)r - 1/2\sigma^2\alpha_i(\alpha_i - 1)$ ,  $i = 1, 2$ , and  $\beta_1$  and  $\beta_2$  are constants chosen to satisfy the boundary conditions, and

$$\alpha_1 = \left(\frac{1}{2} - \frac{r}{\sigma^2}\right) + \frac{\sqrt{(r + 1/2\sigma^2)^2 - 2\gamma\sigma^2}}{\sigma^2}, \quad (3.10)$$

$$\alpha_2 = \left(\frac{1}{2} - \frac{r}{\sigma^2}\right) - \frac{\sqrt{(r + 1/2\sigma^2)^2 - 2\gamma\sigma^2}}{\sigma^2}. \quad (3.11)$$

The boundary conditions are that  $W(S, 0) = W_0$  and  $W(0, T) = F(T) = K$ . The floor  $F(t)$  must grow at the riskfree interest rate. That is,  $F(t) = Ke^{-r(T-t)}$ .

*Proof.* Since the wealth function is of the form (3.8), substituting  $W_t$ ,  $W_S$  and  $W_{SS}$  into (3.3) yields:

$$\left[F'(t) - rF(t)\right] + \left[g'(t) - rg(t)\right]h(S) + rSg(t)h'(S) + \frac{1}{2}\sigma^2 S^2 g(t)h''(S) = 0. \quad (3.12)$$

Assuming that  $g'(t) = \gamma g(t)$  and  $\gamma$  is a constant, (3.14) can be reduced to two ordinary differential equations:

$$F'(t) = rF(t), \quad (3.13)$$

$$\frac{1}{2}\sigma^2 S^2 h''(S) + rSh'(S) + (\gamma - r)h(S) = 0 \quad (3.14)$$

with boundary conditions  $F(T) = K$  and  $h(0) = 0$ .

The solution to the first ODE (3.13) is that  $F(t) = Ke^{-r(T-t)}$ . To solve the second ODE (3.14), we first guess the form  $h(S) = \beta S^\alpha$  and then substitute  $h(S)$ ,  $h'(S)$  and  $h''(S)$  into (3.14). It follows that the general solution to (3.14) is of the form (3.9) and  $\alpha_1$  and  $\alpha_2$  are the roots of the quadratic equation  $1/2\sigma^2\alpha^2 + (r - 1/2\sigma^2)\alpha + (\gamma - r) = 0$ .  $\square$

As a result, the following theorem gives a characterisation of time-invariant portfolio strategies in terms of the proportion of wealth above the floor allocated to the risky asset.

**Theorem 3.2** *The trading strategy for a generalised (borrowing unconstrained) time-invariant portfolio (3.9) in terms of the proportion of wealth above the floor*



allocated to the risky asset is of the form:

$$x(S) = \alpha_1 \theta(S) + \alpha_2 (1 - \theta(S)), \quad (3.15)$$

where

$$\theta(S) = \frac{\beta_1 S^{\alpha_1}}{\beta_1 S^{\alpha_1} + \beta_2 S^{\alpha_2}}. \quad (3.16)$$

*Proof.* This is an immediate result from (3.6) and (3.9).  $\square$ .

The above two theorems are modified versions of Theorem 3 and Theorem 4 in Brennan and Schwartz [14] so as to include the nonzero minimum requirement. Therefore, they provide a complete characterisation of the wealth function and its associated trading rule of a time-invariant portfolio strategy. However, for a portfolio strategy to be efficient, the wealth function must be non-negative and monotonic increasing with respect to the value of the risky asset. By inspection of (3.9), it can be seen that these conditions are not always satisfied. The fact that the contingent claim part of the wealth,  $C(S, t)$ , is a combination of two power terms suggests that the wealth function may not be monotonic increasing and could possibly become negative. Therefore, we need to identify a subclass of strategies where both non-negativity and monotonicity conditions are satisfied.

A possible remedy for these problems is to propose a switching strategy. For the function which exists a minimum value  $C^*$  at  $S^*$ , the strategy is to switch all the risky holding to the riskless asset when the risky asset price falls below  $S^*$ . This assures the function to be monotonically increasing. However, these strategies are no longer path-independent and are therefore inefficient.



### 3.3 Characterisation of Efficient Strategies

In the previous section, we provided a general solution of the wealth function generated by time-invariant strategies. However, as discussed before, for a strategy to be efficient, it must satisfy three conditions: path-independence, non-negativity and monotonicity. Therefore, in this section, we shall use these criteria to identify and characterise efficient strategies amongst the type of time-invariant strategies described in Theorem 3.1 and Theorem 3.2.

**Proposition 3.1** *In a Black-Scholes economy where  $r$  and  $\sigma$  are constant, the wealth function of an efficient time-invariant portfolio strategy is characterised as follows:*

(P1) *If  $r \geq \sigma^2/2$ ,  $\alpha \geq 0$ , or  $0 < r < \sigma^2/2$ ,  $\alpha > 1 - 2r/\sigma^2$ , the efficient wealth function is given by:*

$$W(S, t) = Ke^{-r(T-t)} + (W_0 - Ke^{-rT})e^{\gamma t} \left( \frac{S_t}{S_0} \right)^\alpha, \quad (3.17)$$

where  $W_0$ ,  $K$ , and  $\alpha = \alpha_1$  are the same as defined in Theorem 3.1,  $\gamma = (1 - \alpha)r - \frac{1}{2}\alpha(\alpha - 1)\sigma^2$ , and  $S_0$  and  $S_t$  are values of the non-dividend-paying risky asset at time 0 and time  $t$ , respectively. The investment strategy is to invest a constant (nonnegative) proportion  $\alpha$  of wealth above the floor in the risky asset.

(P2) *If  $0 < r < \sigma^2/2$  and  $0 \leq \alpha_1 \leq 1 - 2r/\sigma^2$ , the efficient wealth function is given by:*

$$W(S, t) = Ke^{-r(T-t)} + (W_0 - Ke^{-rT})e^{\gamma t} \left[ c \left( \frac{S_t}{S_0} \right)^{\alpha_1} + (1 - c) \left( \frac{S_t}{S_0} \right)^{\alpha_2} \right], \quad (3.18)$$

where  $W_0$ ,  $K$ ,  $\alpha_1$ , and  $\alpha_2$  are the same as defined in Theorem 3.1,  $0 \leq c \leq 1$  is a constant, and  $S_0$  and  $S_t$  are values of the non-dividend-paying risky asset at time

0 and time  $t$ , respectively. The investment strategy is to invest a proportion  $x(S)$  of wealth above the floor in the risky asset, where

$$x(S) = \alpha_1 f(S) + \alpha_2 [1 - f(S)], \quad (3.19)$$

$$f(S) = \frac{c (S/S_0)^{\alpha_1}}{c (S/S_0)^{\alpha_1} + (1 - c) (S/S_0)^{\alpha_2}}. \quad (3.20)$$

*Proof.* In order for the wealth function (3.9) to be efficient, the following two conditions must be satisfied. First, the cushion must be nonnegative at least at time  $T$ . This is a direct result from the realisation of  $K$  being the minimum value at  $T$ . Namely,  $W(S, T) \geq K$ , and  $C(S, T) \geq 0$ . Second, the wealth function must be monotonically increasing with respect to  $S$ . That is,  $W_S \geq 0$ . More specifically, the efficient strategies must satisfy the following two inequalities:

$$\beta_1 S^{\alpha_1} + \beta_2 S^{\alpha_2} \geq 0, \quad (3.21)$$

$$\alpha_1 \beta_1 S^{\alpha_1-1} + \alpha_2 \beta_2 S^{\alpha_2-1} \geq 0. \quad (3.22)$$

By inspection of the general solution (3.9), We can rule out the situation where both  $\alpha_1$  and  $\alpha_2$  are negative, since it would result in a monotonically decreasing wealth function which is obviously inefficient. Similarly, when either  $\alpha_1$  or  $\alpha_2$  is negative, the wealth function will appear to be non-monotonic and the minimum wealth  $W^*$  occurs when  $S_T^* \in (0, \infty)$ . Again, it would be inefficient and, therefore, the coefficient of the negative root must be zero for the wealth function to be monotonically increasing.

From (3.10) and (3.11), it can be seen that  $\alpha_2 = 1 - 2r/\sigma^2 - \alpha_1$  (or vice versa). Hence, there are two possibilities: (i) when  $\alpha_1 \geq 1 - 2r/\sigma^2$ ,  $\alpha_2 \leq 0$ , or (ii) when  $\alpha_1 \leq 1 - 2r/\sigma^2$ ,  $\alpha_2 \geq 0$ . It is convenient to fix a non-negative  $\alpha_1$  first, and then



solve for  $\alpha_2$ . Therefore, under (i),  $\alpha_2$  is always less than 0 except when  $r = \sigma^2/2$ ,  $\alpha_1 = 0$ , and  $\alpha_2 = 0$ . By setting  $\beta_2 = 0$ , we can be rid of a negative  $\alpha_2$ . Equation (3.17) then follows immediately after matching the boundary conditions. Since  $\beta_2 = 0$ , from (3.15) and (3.16), it is also immediate that the strategy is a constant one, i.e.,  $x = \alpha_1$ .

Finally, given  $\alpha_1 \geq 0$  and (ii), it is straightforward to show that when  $0 < r < \sigma^2/2$ , both  $\alpha_1$  and  $\alpha_2$  are non-negative when  $0 \leq \alpha_1 \leq 1 - 2r/\sigma^2$ , and  $0 \leq \alpha_2 \leq 1 - 2r/\sigma^2$ . In order to satisfy the necessary conditions (3.21) and (3.22), both  $\beta_1$  and  $\beta_2$  must be non-negative as well. Again, (3.18) follows immediately after matching the boundary conditions. Equations (3.19) and (3.20) are alternative expressions of (3.15) and (3.16) for the strategy when setting the constant  $c = \beta_1 S_0^{\alpha_1} / (\beta_1 S_0^{\alpha_1} + \beta_2 S_0^{\alpha_2})$ . Since  $\beta_1 \geq 0$  and  $\beta_2 \geq 0$ ,  $0 \leq c \leq 1$ .  $\square$

**Remark 3.1** *Brennan and Schwartz's [14] characterisation of time-invariant portfolio insurance strategies is in fact incomplete since it does not encompass the case where a non-zero minimum guarantee is required. Moreover, in their work [14], it is not clear exactly under what theoretical constraints the time-invariant portfolio strategies will be efficient. Although the paper was published almost one and a half decades ago, the problems therein have not been formally addressed and corrected in the literature. Therefore, one of the main contributions of this chapter is to clarify this point and provide a correct version so as to fill the gap in the literature.*

Proposition 3.1 suggests a richer set of efficient strategies than expected. According to our definition, there are two possible functional forms for the wealth of



an efficient time-invariant portfolio strategy: one results from a *constant policy* and the other from a *level-dependent* policy. It is interesting to analyse the behaviour of the wealth function in more detail. For the constant policy, the parameter  $\alpha$  not only determines the growth rate of the cushion  $\gamma$  but also determines the shape of the payoff function. Specifically, if  $\alpha = 0$ , then  $\gamma = r$  and  $W_S = 0$ . This corresponds to an extremely risk-averse investor who demands a minimum value  $K = W_0 e^{rT}$  so that the optimal strategy is to invest in the riskless asset only. If  $\alpha = 1$ , then  $\gamma = 0$  and  $W_S$  is a constant. This corresponds to investing 100% of the cushion in the risky asset, or a buy-and-hold strategy that will create a linear payoff. If  $0 < \alpha < 1$ , then  $0 < \gamma < r$  and  $W_{SS} < 0$ . This then corresponds to a contrarian strategy that generates a concave payoff. Finally, if  $\alpha > 1$ , then  $\gamma < 0$  and  $W_{SS} > 0$ . This corresponds to a portfolio insurance strategy that gives a convex payoff. In fact, this convex type of strategy has been popularised and termed the *constant proportion portfolio insurance* (CPPI) policy, mainly by Black and Jones [7, 8] and Black and Perold [9].

For the level-dependent strategy, the non-trivial case is when  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$ , and  $\beta_2$  are all positive. Since also  $\alpha_1 + \alpha_2 = 1 - 2r/\sigma^2 < 1$ , we have  $0 < \alpha_1 < 1$  and  $0 < \alpha_2 < 1$ . It can then be shown that  $W_{SS} < 0$ . That is, it generates a concave payoff. We shall refer to this type of strategy as the *level-dependent contrarian strategy* in order to distinguish from the *constant proportion contrarian strategy*.

Finally, before we close this section, we would like to provide a result which relates the constant proportion portfolio strategy to the investor's preference. The following proposition shows that the constant policy can be rationalised in the

context of expected utility maximisation and that it is an optimal strategy for an investor with a *translated* isoelastic utility function.

**Proposition 3.2** *Without borrowing restrictions and intermediate consumption, the optimal portfolio investment strategy for an investor facing a constant opportunity set and maximising her expected utility (a translated isoelastic utility function with a constant RRA coefficient  $\lambda > 0$ ) over the terminal wealth is to invest a constant proportion,  $\frac{\mu-r}{\sigma^2\lambda}$ , of wealth above the (possibly non-zero) floor in the non-dividend-paying risky asset during the investment horizon.*

*Proof.* Choose the optimal control  $\alpha$ , the fraction of wealth above the floor to be invested in the non-dividend-paying risky asset, to maximise the expected utility over terminal wealth:

$$\max_{\alpha} \mathbb{E}U(W_T) \quad (3.23)$$

subject to

$$\begin{aligned} \frac{dS}{S} &= \mu dt + \sigma dz, \\ dW &= (W - \alpha C)r dt + \alpha C \left( \frac{dS}{S} \right), \\ U(W_T) &= \begin{cases} \frac{(W_T - K)^{1-\lambda}}{1-\lambda} & \text{for } \lambda > 0, \lambda \neq 1 \\ \ln(W_T - K) & \text{for } \lambda = 1. \end{cases} \end{aligned}$$

where  $K \leq W_T$  is the predetermined minimum wealth level at  $T$ ,  $C(t) = W(t) - Ke^{-r(T-t)}$  is the *cushion*, and  $\lambda$  is the constant relative risk aversion coefficient.

The value function is defined as  $J = \max \mathbb{E}U(W_T)$ . The *Bellman equation* is then given by

$$0 = J_t + J_W [(W - \alpha C)r + \alpha C\mu] + \frac{1}{2} J_{WW} \alpha^2 C^2 \sigma^2. \quad (3.24)$$

It is well known that the optimal solution  $\alpha^*$  to the problem is as follows:

$$\alpha^* = \frac{\mu - r}{\sigma^2 \left( -C \frac{J_{WW}}{J_W} \right)} = \frac{\mu - r}{\sigma^2 \lambda}. \quad (3.25)$$

□

### 3.4 Numerical Examples

In this section, we visualise the main results of our model, Theorem 3.1 and Proposition 3.1 by applying realistic parameter values to the relevant functions.

For convenience, the initial index  $S_0$  is set as 100 and the initial wealth  $W_0$  is set as equal to 100 as well. For the floor (the minimum level of wealth), instead of setting  $F(T) = K$ , we fix the initial value  $F(0) = F_0 = 80$ . This is simply for convenience and will not have any effect on our model. We also fix the investment horizon as 10 years.

The set of parameters present in the model is given as follows. The volatility  $\sigma$  is 0.2 and the interest rate  $r$  is equal to 6% ( $> \sigma^2/2$ ), 2% ( $= \sigma^2/2$ ), or 1% ( $< \sigma^2/2$ ).  $\alpha$  in (3.17) and  $\alpha_1$ ,  $\alpha_2$  and  $c$  in (3.18) satisfy the following conditions:

$$\alpha \geq 0,$$

$$\gamma = (1 - \alpha)r - 1/2\sigma^2\alpha(\alpha - 1),$$

$$\alpha_1 + \alpha_2 = 1 - 2r/\sigma^2.$$

$$0 \leq c \leq 1.$$

Table 2.1 tabulates the coefficients of the model (3.9) for different combinations of  $\alpha_1$ ,  $\alpha_2$  and  $\gamma$ . It can be seen that there are mainly two cases: (1) a non-negative



Table 3.1: **Coefficients of the time-invariant portfolio strategy.** This table shows the coefficients of the model (3.9) for  $r = 6\%(> \sigma^2/2)$ ,  $2\%(= \sigma^2/2)$ ,  $1\%(< \sigma^2/2)$ . The value  $\alpha_1$  is set to vary from 0 to 5, while  $\alpha_2$  and  $\gamma$  are calculated from  $\alpha_2 = 1 - 2r/\sigma^2 - \alpha_1$  and  $\gamma = (1 - \alpha_1)r - 1/2\sigma^2\alpha_1(\alpha_1 - 1)$ . The initial index  $S_0$  is 100, the initial wealth  $W_0$  is 100, and the initial floor  $F_0$  is 80. The volatility  $\sigma$  is 0.2 and the investment horizon  $T$  is 10 years.

$\alpha_1$	$r = 0.06$		$r = 0.02$		$r = 0.01$	
	$\alpha_2$	$\gamma$	$\alpha_2$	$\gamma$	$\alpha_2$	$\gamma$
0.0	-2.0	0.0600	0.0	0.0200	0.5	0.0100
0.1	-2.1	0.0558	-0.1	0.0198	0.4	0.0108
0.2	-2.2	0.0512	-0.2	0.0192	0.3	0.0112
0.3	-2.3	0.0462	-0.3	0.0182	0.2	0.0112
0.4	-2.4	0.0408	-0.4	0.0168	0.1	0.0108
0.5	-2.5	0.0350	-0.5	0.0150	0.0	0.0100
0.6	-2.6	0.0288	-0.6	0.0128	-0.1	0.0088
0.7	-2.7	0.0222	-0.7	0.0102	-0.2	0.0072
0.8	-2.8	0.0152	-0.8	0.0072	-0.3	0.0052
0.9	-2.9	0.0078	-0.9	0.0038	-0.4	0.0028
1.0	-3.0	0.0000	-1.0	0.0000	-0.5	0.0000
1.5	-3.5	-0.0450	-1.5	-0.0250	-1.0	-0.0200
2.0	-4.0	-0.1000	-2.0	-0.0600	-1.5	-0.0500
2.5	-4.5	-0.1650	-2.5	-0.1050	-2.0	-0.0900
3.0	-5.0	-0.2400	-3.0	-0.1600	-2.5	-0.1400
3.5	-5.5	-0.3250	-3.5	-0.2250	-3.0	-0.2000
4.0	-6.0	-0.4200	-4.0	-0.3000	-3.5	-0.2700
4.5	-6.5	-0.5250	-4.5	-0.3850	-4.0	-0.3500
5.0	-7.0	-0.6400	-5.0	-0.4800	-4.5	-0.4400



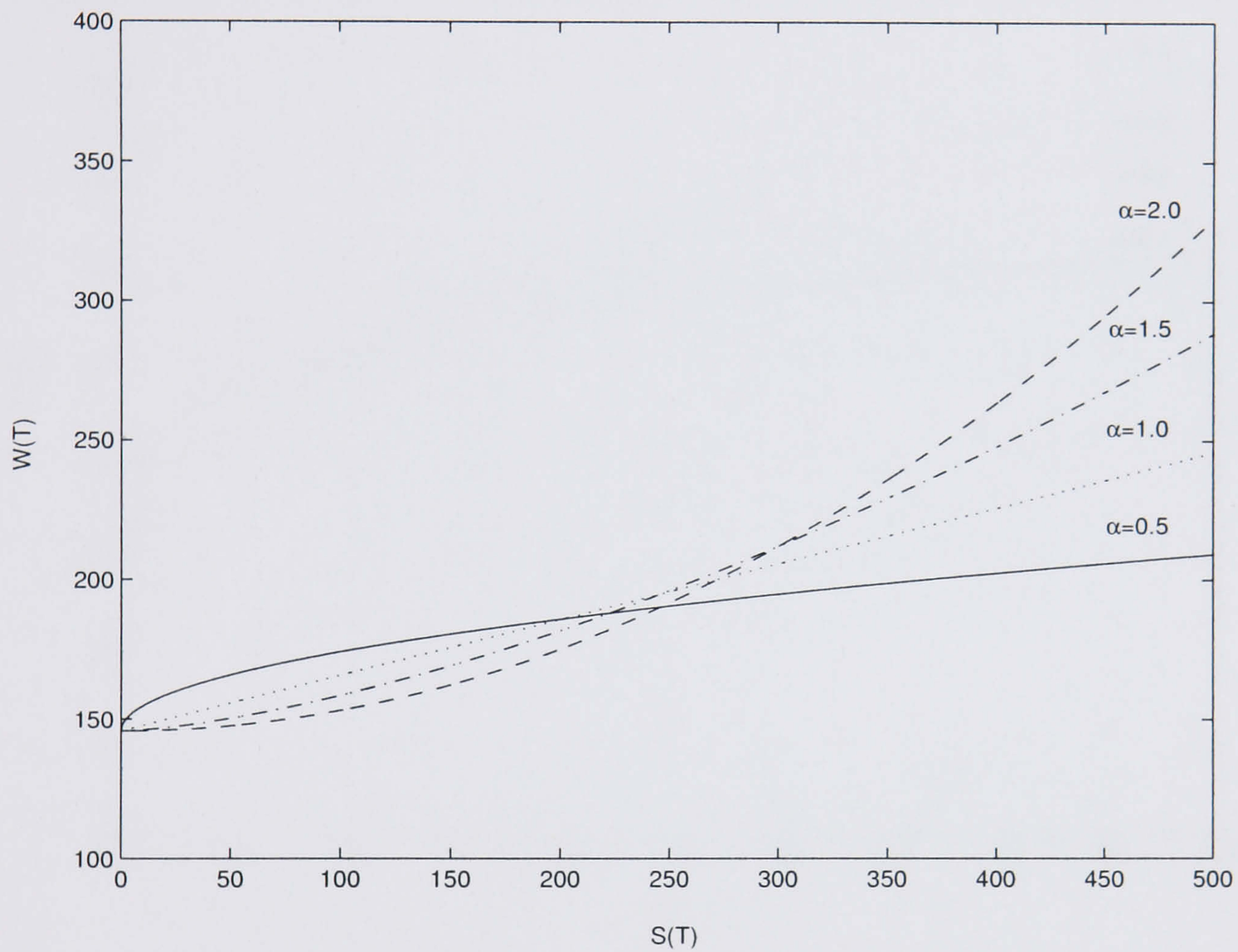


Figure 3.1: **Payoff functions of the efficient constant proportion portfolio strategy:**  $r > \sigma^2/2$ . This figure shows the behaviour of the model (3.17) for  $\alpha_1 = 0.5, 1.0, 1.5, 2.0$ . The riskless interest rate  $r$  is 0.06, the volatility  $\sigma$  is 0.2, and the investment horizon  $T$  is 10 years. The initial index  $S_0$  is 100, the initial wealth  $W_0$  is 100, and the initial floor  $F_0$  is 80.



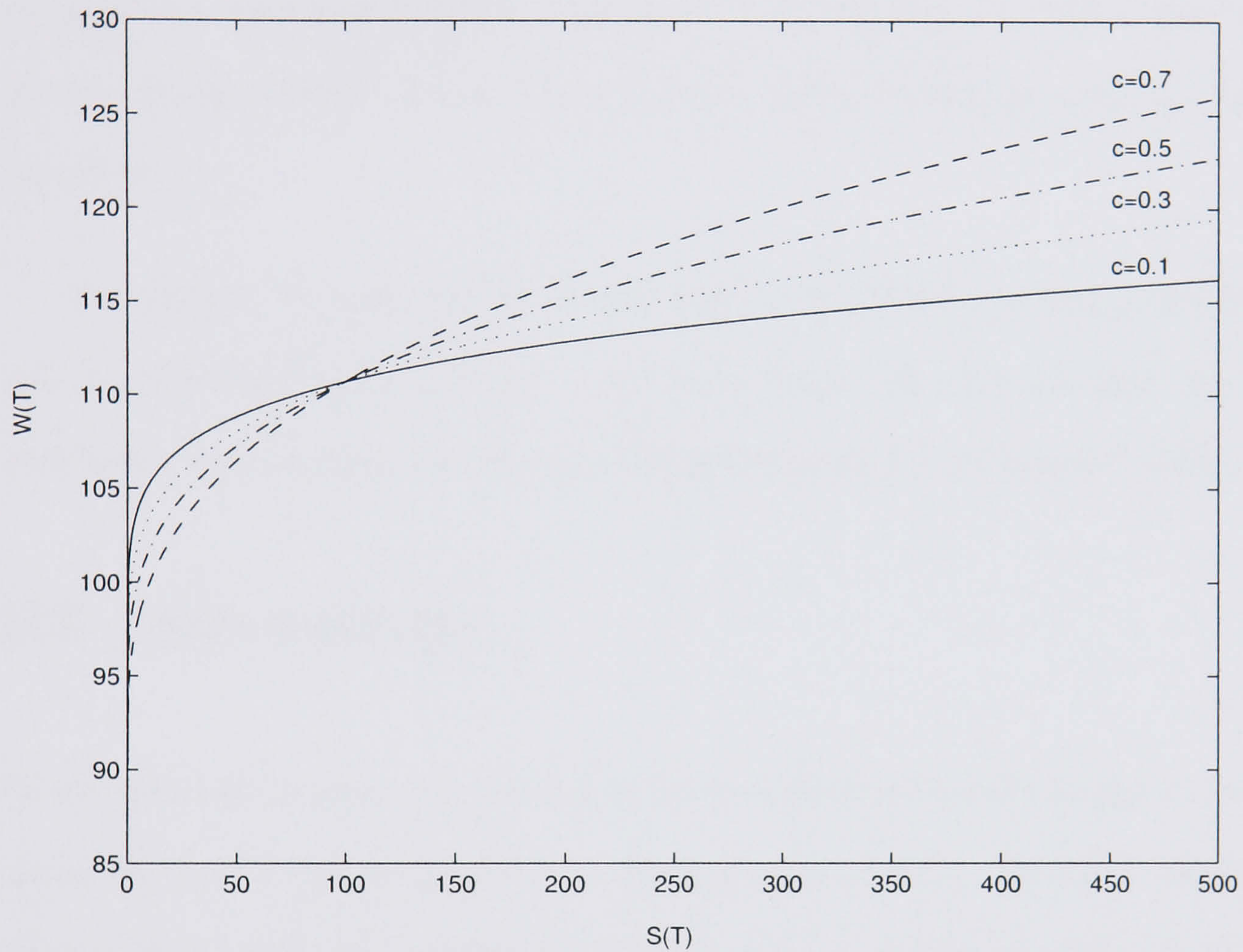


Figure 3.2: **Payoff functions of the efficient level-dependent contrarian strategy:**  $0 < r < \sigma^2/2$ . This figure shows the behaviour of the model (3.18) for  $c = 0.1, 0.3, 0.5, 0.7$ . The set of parameter values used are:  $\alpha_1 = 0.4$ ,  $\alpha_2 = 0.1$ , and  $\gamma = 0.0108$ . The interest rate  $r$  is 0.01, the volatility  $\sigma$  is 0.2, and the investment horizon  $T$  is 10 years. The initial index  $S_0$  is 100, the initial wealth  $W_0$  is 100, and the initial floor  $F_0$  is 80.



$\alpha_1$  with a non-positive  $\alpha_2$ ; (2) a non-negative  $\alpha_1$  with a non-negative  $\alpha_2$ . The efficient function for the first case is the constant policy (3.17) and payoff is shown in Figure 2.1 for  $\alpha = 0.5, 1.0, 1.5, 2.0$ . It is interesting to note that when  $\alpha$  is a fraction, the wealth function corresponds to a contrarian strategy. It will dominate the portfolio insurance strategy (when  $\alpha \geq 1$ ) for the region of lower risky asset prices. On the contrary, the portfolio insurers will be compensated by the upside growth.

The efficient function for the second case is the level-dependent policy (3.18) and is shown in Figure 3.2 for  $c = 0.1, 0.3, 0.5, 0.7$ . We can see that this is a contrarian type strategy and the level of capital growth is, in general, fairly low.

## 3.5 Discussions

In our previous analysis, we noted that for the time-invariant strategy to be efficient, the floor  $F(t)$  must grow at the riskless interest rate  $r$ . We either determine the minimum value  $K$  by letting  $F(T) = K \leq W_0 e^{rT}$ , or fix the initial floor  $F(0)$  by letting  $F(0) \leq W_0$ . For an indefinite investment horizon, it suffices since we can do the latter. However, the fact that our *time-invariant* strategy contains a *time-varying* floor may be a concern. Nevertheless, we will show that this is not a problem at all.

If we use the bank account as the numeraire, then we can denote the normalised value of the risky asset by  $\hat{S}(t) = S(t)/B(t)$ , and the value of the riskless asset will become a constant one. This is equivalent to let the interest rate  $r$  be zero.

Therefore, the floor will become a constant. we can then implement the same time-invariant strategy in a normalised market, where the normalised wealth  $\hat{W}(\hat{S}, t) = W(S, t)/B(t)$  will follow the same forms as in (3.17) and (3.18) with  $r = 0$  and  $S$  and  $W$  replaced by  $\hat{S}$  and  $\hat{W}$ , respectively.

Now, recall that we defined a time-invariant strategy in terms of the proportion of wealth above the floor being time-independent. An alternative definition may be to define a time-invariant strategy in terms of the wealth itself being independent of time. This means that  $W$  is a function of  $S$  only. This property has a close relationship with that of a perpetual American option. Indeed, Black and Perold [9] showed that the payoff of a constant policy can be achieved by investing in perpetual American call options. However, to build this equivalence, it is necessary to impose borrowing constraints on the constant policy. When the borrowing constraint is binding, it resembles the possible early exercise in the options, and when the borrowing constraint becomes unbinding *again*, it resembles the reinvestment in the options.

Moreover, it is well known that perpetual American call options must be written on dividend-paying securities. Therefore, the constant policy should also be applied to a dividend-paying risky asset, and dividend payouts should be reinvested. The important feature of introducing borrowing constraints and dividend is that both the constant policy and the dynamic strategy in perpetual American call options will become path-dependent (hence dominated strategies), unless the dividend is consumed rather than reinvested. Thus, the constant policy can be



shown to be utility maximising when intertemporal consumption is introduced.<sup>2</sup>

## 3.6 Chapter Summary and Conclusions

In this chapter, we first examined, in a perfect market setting, the general form of wealth functions for time-invariant portfolio strategies with possible non-zero minimum requirements. In light of conditions for efficient portfolio strategies, we then provided a correct characterisation of efficient time-invariant strategies. It turned out that, under our definition of time-invariance of the strategy, the only efficient ones are: (1) a constant policy; and (2) a level-dependent policy. Our analysis resolves the problem in Brennan and Schwartz [14] where the minimum requirement was not taken into account. Our model also provides a much richer behaviour than the continuous-time frictionless representation of the constant proportion portfolio insurance by Black and Perold [9] in that our constant policy generates not only convex payoffs, but also linear or concave ones.

Time-invariant portfolio strategies are of interest. However, it can be argued that the dynamic asset allocation problem would become much more complex if the investor's opportunity set is time-varying. A natural extension is to relax the assumption of a constant interest rate in the Black-Scholes world and to assume for example a stochastic interest rate. The dynamic asset allocation problem under stochastic interest rates will be the subject of the next chapter.

Another possible extension can also be made along the line of considering a

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<sup>2</sup>See Black and Perold [9] for detailed analysis on this account.



time-varying risk premium. In particular, earlier work by He and Leland [57] and Hodges and Carverhill [61] suggest that in equilibrium the risk premium becomes more sensible when approaching investment horizon. This finding also implies that in an overlapping generations economy, long horizon investors will be contrarian and short horizon ones will portfolio insure. Risk/return trade-offs must be consistent through time, but path-independence property may no longer apply. The evolution of the market risk premium will be explored in Chapter 5.

# Chapter 4

## Dynamic Asset Allocation under Stochastic Interest Rates

In this chapter, we explore a three-asset allocation problem in a complete market setting. The assumption of the constancy of interest rate in the Black-Scholes model is relaxed. Instead, the interest rates are assumed to be stochastic. The factor risk premium is determined so as to be consistent with the underlying equilibrium. The dual approach is then applied to solve explicitly for the individual investor's optimal portfolio choice under the assumption of a Vasicek short rate model [103] and alternative utility functions.

### 4.1 Introduction

Conventionally, dynamic asset allocation problems are solved by an explicit specification of the investor's utility function and the opportunity set using the arbitrage approach. One of the potential problems of the arbitrage approach is that not every choice of asset price process is viable. In other words, an exogenously specified

opportunity set may not be supported by an underlying equilibrium.<sup>1</sup>

In this chapter, we propose a framework where the factor risk premia can be determined in such a way that they will be consistent with equilibrium. We then apply the equilibrium factor risk premium to solve explicitly for the dynamic asset allocation problem of a non-representative agent. Our equilibrium is described by a representative agent who maximises the expected utility of consumption at the horizon date. The utility function is of a von Neumann-Morgenstern type. The methodology is similar to that of He [57].

A significant assumption made in our model is that we allow a state-dependent utility function - the utility function depends on level of wealth and the interest rate at the horizon date. The state-dependent utility assumption can be found in He [57] where the volatility of the stock returns is determined by a stochastic state variable and the utility function of the representative agent depends on the final levels of wealth and the state variable. As we shall see later, this specification allows a much richer class of factor risk premium functions for the state variable. In our example of a Vasicek short rate model, it is shown that a zero factor risk premium is supported by a state-independent log utility and a constant factor risk premium is supported by a state-dependent power utility. It is also shown that when the representative agent's utility is state-dependent, the utility is increasing in the interest rate, i.e., the agent derives more utility when the interest rate is high.

In deriving the non-representative agent's optimal portfolio allocation, two

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<sup>1</sup>See Cox, Ingersoll and Ross. [28].



types of utility functions are assumed. First, we consider a CRRA utility function of terminal wealth (ie, the utility function is independent of interest rate). Then we consider the situation where the non-representative agent has a utility function of the same form as the representative agent, but different coefficient of relative risk aversion. In this case, the utility function is dependent upon the interest rate. Under the assumption of a constant equity premium and a Vasicek type of short rate process [103], we show that the optimal choice in the market index is a constant one but the optimal choice in the discount bond is such that the proportion should increase as the non-representative agent becomes more risk averse.

The remainder of the chapter is organised as follows. Section 4.2 describes the basic setting of our model and constructs the optimal control problem of the investor. Section 4.3 then characterises the equilibrium asset price process and the factor risk premium function in particular. Section 4.4 applies the analysis to derive the supporting utility under the Vasicek short rate assumption. The dynamic asset allocation problem for this setting is solved explicitly in Section 4.5, using the dual approach. Section 4.6 concludes the chapter.

## 4.2 The Optimal Portfolio Choice Problem

Consider, in a complete market setting, an investor's optimal portfolio choice problem. The investor aims to maximise her expected utility over wealth at terminal date and can trade three types of assets: a stock index, a pure discount bond maturing at time  $T$  (or a  $T$ -bond), and a bank account which accumulates at the short rate of interest  $r$ . The investor's optimisation problem will be properly

defined and solved later, after we have described the financial markets and the financial assets therein.

The stock represents the market portfolio and its value  $S$  follows a diffusion process

$$\frac{dS_t}{S_t} = \mu(S, r, t)dt + \sigma(S, r, t)dz_{1,t}, \quad (4.1)$$

where  $\mu$  and  $\sigma$  are functions of  $S$ ,  $r$  and  $t$ . The short rate of interest  $r$  follows a one-factor stochastic process

$$dr_t = \alpha(r, t)dt + \beta(r, t) \left( \rho dz_{1,t} + \sqrt{1 - \rho^2} dz_{2,t} \right), \quad (4.2)$$

where  $z_1$  and  $z_2$  are standard Brownian motions under the objective probability measure  $\mathbb{P}$  and  $\rho$  is the correlation coefficient between  $dz_1$  and  $dz_2$ .

We further assume that the economy does not permit arbitrage. Hence, it is well known that there must exist a unique equivalent martingale measure  $\mathbb{Q}$  under which the processes (4.1) and (4.2) can be transformed to the following processes

$$\frac{dS_t}{S_t} = r(t)dt + \sigma(S, r, t)d\tilde{z}_{1,t} \quad (4.3)$$

and

$$dr_t = (\alpha(r, t) - \Lambda(S, r, t))dt + \beta(r, t) \left( \rho d\tilde{z}_1 + \sqrt{1 - \rho^2} d\tilde{z}_2 \right), \quad (4.4)$$

respectively, where  $\tilde{z}_1$  and  $\tilde{z}_2$  are Brownian motions under the  $\mathbb{Q}$ -measure, i.e.,

$$\begin{aligned} d\tilde{z}_1 &= dz_1 + \frac{\mu(S, r, t) - r(t)}{\sigma(S, r, t)}dt \\ d\tilde{z}_2 &= dz_2 + \nu(S, r, t)dt \end{aligned}$$

and

$$\Lambda(S, r, t) = \left( \frac{\mu(S, r, t) - r(t)}{\sigma(S, r, t)} \rho + \nu(S, r, t) \sqrt{1 - \rho^2} \right) \beta(r, t)$$

is the risk premium for the interest rate and is the covariance of changes in the interest rate with proportional changes in the market portfolio. Since the impact of the arbitrary function  $\nu$  is through the above equation, we shall call  $\nu$  the *factor risk premium function* throughout the analysis.

Let us denote by  $P(t, T)$  the T-bond and by  $N(t)$  the current value of the bank account. One unit of the discount bond pays off £1 at the investor's planning horizon  $T$ . The bank account starts with 1 at time 0. Furthermore, denote by  $X_S$ ,  $X_P$ , and  $X_N$ , the units invested in the stock index, the bond, and the bank account, respectively. Thus, the investor's wealth at time  $t$  can be written as

$$W(t) = X_S S(t) + X_P P(t, T) + X_N N(t). \quad (4.5)$$

From the *Local Expectations Hypothesis*, the time  $t$  price of the  $T$ -bond,  $P(t, T)$ , can be written as

$$P(t, T) = \mathbb{E}_t^Q \left[ \exp \left( - \int_t^T r(u) du \right) \right],$$

where the expectation is taken under the  $\mathbb{Q}$ -measure.

Therefore, by Itô's lemma, we can derive the process for the bond price  $P(t, T) \equiv P(r, t; T)$  under the  $\mathbb{P}$ -measure:

$$\frac{dP}{P} = \frac{1}{P} \left( P_t + \alpha P_r + \frac{1}{2} \beta^2 P_{rr} \right) dt + \frac{1}{P} P_r \beta \left( \rho dz_1 + \sqrt{1 - \rho^2} dz_2 \right) \quad (4.6)$$

and

$$\frac{dP}{P} = \frac{1}{P} \left[ P_t + (\alpha - \Lambda) P_r + \frac{1}{2} \beta^2 P_{rr} \right] dt + \frac{1}{P} P_r \beta \left( \rho d\tilde{z}_1 + \sqrt{1 - \rho^2} d\tilde{z}_2 \right) \quad (4.7)$$

under the  $\mathbb{Q}$ -measure. Since the drift rate in (4.7) must be equal to the short rate,



$P(r, t; T)$  must satisfy the following partial differential equation

$$P_t + (\alpha - \Lambda)P_r + \frac{1}{2}\beta^2 P_{rr} - rP = 0 \quad (4.8)$$

subject to the boundary condition  $P(r, T; T) = 1$ .

Substituting (4.8) into (4.6) yields

$$\frac{dP}{P} = (r - \Lambda D)dt - \beta D \left( \rho dz_1 + \sqrt{1 - \rho^2} dz_2 \right), \quad (4.9)$$

where  $D \equiv D(r, t; T) = -P_r/P$  denotes the duration of the discount bond and is also the elasticity of the bond price with respect to the short interest rate. The risk premium for the bond is equal to  $-\Lambda D$ . Note that for ease of exposition, we sometimes employ shorthand notations for  $\mu(S, r, t)$ ,  $\sigma(S, r, t)$ ,  $r(t)$ ,  $\nu(S, r, t)$ ,  $\Lambda(S, r, t)$ ,  $D(r, t; T)$  in order to make the equations more readable.

Let  $\phi_S$  and  $\phi_P$  denote the proportions of wealth invested in the stock and the bond, respectively. The instantaneous proportional change in wealth can, therefore, be written as

$$\begin{aligned} \frac{dW}{W} = & [r + \phi_S(\mu - r) + \phi_P(-\Lambda D)] dt \\ & + \phi_S \sigma dz_1 - \phi_P \beta D \left( \rho dz_1 + \sqrt{1 - \rho^2} dz_2 \right), \end{aligned} \quad (4.10)$$

The investor's problem is now to allocate the wealth among the three available assets in order to maximise the expected utility over wealth at the horizon date:

$$\max_{\phi_S, \phi_P} \mathbb{E}[U(r_T, W_T)] \quad (4.11)$$

subject to (4.10) and the budget constraint  $W(0) = W_0 > 0$ .  $U$  is a strictly increasing and continuously differentiable von Neumann-Morgenstern utility function. Note that by specifying  $U \equiv U(r_T, W_T)$ , we have deliberately formulated the

possible state-dependency on the state variables in the utility function. Since there are two state variables  $S$  and  $r$  in our setting, following the standard procedures of stochastic dynamic programming, we can define the indirect utility function  $J(W, S, r, t)$  with the boundary condition  $J(W, S, r, T) = U(r_T, W_T)$ . Thus, the Hamilton-Jacobi-Bellman equation can be written as

$$\begin{aligned}
0 = & \max_{\phi_S, \phi_P} \{ J_t + J_W W [r + \phi_S(\mu - r) + \phi_P(-\Lambda D)] \\
& + \mu S J_S + \alpha J_r + \frac{1}{2} W^2 J_{WW} (\phi_S^2 \sigma^2 + \phi_P^2 \beta^2 D^2 - 2\phi_S \phi_P \rho \sigma \beta D) \\
& + \frac{1}{2} \sigma^2 S^2 J_{SS} + \frac{1}{2} \beta^2 J_{rr} + W S J_{WS} (\sigma^2 \phi_S - \rho \sigma \beta D \phi_P) \\
& + J_{Wr} W (\rho \sigma \beta \phi_S - \beta^2 D \phi_P) + \rho \sigma \beta S J_{Sr} \}. \tag{4.12}
\end{aligned}$$

Solving the system of equations from the first order conditions yields the optimal controls:

$$\phi_S^* = \left( \frac{\mu - r}{\sigma^2} - \frac{\rho}{\sigma \sqrt{1 - \rho^2}} \nu \right) \left( -\frac{J_W}{W J_{WW}} \right) - \frac{S J_{WS}}{W J_{WW}}, \tag{4.13}$$

$$\phi_P^* = \left( -\frac{1}{\sqrt{1 - \rho^2}} \frac{\nu}{\beta D} \right) \left( -\frac{J_W}{W J_{WW}} \right) + \frac{1}{DW} \frac{J_{Wr}}{J_{WW}}. \tag{4.14}$$

Since by construction, the strategy is self-financing, the proportion of wealth invested in the bank account is simply  $\phi_R^* = 1 - \phi_S^* - \phi_P^*$ .

### 4.3 Equilibrium Asset Price Dynamics

In the previous section, we have expressed explicitly the individual investor's optimal portfolio choice in terms of partial derivatives of the indirect utility. In this section, we use this result to analyse the equilibrium asset price processes. The kind of equilibrium we employ here is the conventional single representative agent

equilibrium. We first give representations of the equilibrium factor risk premium function and the market price of risk in stock index, and then move to obtain equilibrium conditions for the coefficients of the asset price processes to satisfy in terms of a set of partial differential equations. We will see that with further assumptions, it is possible to back out the representative agent's utility function that support the system.

### 4.3.1 Equilibrium Factor Risk Premium Function

Recall that in our setting, the stock index represents the market portfolio. Thus, we define the market equilibrium in such a way that the representative agent must optimally hold the stock index at all times. By normalising the market portfolio to one share of the stock index, it is required that, in equilibrium, the representative agent holds the market portfolio, i.e.,  $\phi_S^* = 1$  and the bond is in zero net supply, i.e.,  $\phi_P^* = 0$ . By substituting these conditions into (4.13) and (4.14), we can obtain the equilibrium factor risk premium function  $\nu$

$$\nu = -\beta\sqrt{1-\rho^2}\frac{J_{Wr}}{J_W}, \quad (4.15)$$

and the market price of risk for the stock index is

$$\frac{\mu-r}{\sigma} = -\sigma S \frac{J_{WW}}{J_W} - \sigma S \frac{J_{WS}}{J_W} - \rho\beta \frac{J_{Wr}}{J_W}. \quad (4.16)$$



### 4.3.2 Equilibrium Conditions

Assume  $J$  is a smooth function and is differentiable wherever applicable. We can apply *Ito's Lemma* on  $dJ_W$  to yield the following equation,

$$\begin{aligned} dJ_W &= J_{Wt}dt + J_{WW}dW + J_{WS}dS + J_rdr \\ &\quad + \frac{1}{2}J_{WWW}(dW)^2 + \frac{1}{2}J_{WSS}(dS)^2 + \frac{1}{2}J_{Wrr}(dr)^2 \\ &\quad + J_{WWS}(dW)(dS) + J_{WWr}(dW)(dr) \\ &\quad + J_{WSr}(dS)(dr). \end{aligned} \quad (4.17)$$

Due to the appearance of the higher order derivatives, we further differentiate (4.12) with respect to  $W$  and substitute it together with (4.1), (4.2), and (4.10) into (4.17). This gives

$$\begin{aligned} dJ_W &= -rJ_Wdt + WJ_{WW} \left[ \phi_S\sigma dz_1 - \phi_P\beta D \left( \rho dz_1 + \sqrt{1-\rho^2}dz_2 \right) \right] \\ &\quad + J_{WS}\sigma Sdz_1 + J_{Wr}\beta \left( \rho dz_1 + \sqrt{1-\rho^2}dz_2 \right). \end{aligned} \quad (4.18)$$

Finally, substituting (4.13) and (4.14) into (4.18) yields

$$\frac{dJ_W}{J_W} = -rdt - \left( \frac{\mu - r}{\sigma} \right) dz_1 - \nu dz_2. \quad (4.19)$$

By stochastic integration,  $J_W$  has the solution of the following form

$$J_W(t) = J_W(0) \cdot \xi_t^{(\nu)}, \quad (4.20)$$

where

$$\begin{aligned} \xi_t^{(\nu)} &= \exp \left\{ - \int_0^t r_u du - \frac{1}{2} \int_0^t \left[ \left( \frac{\mu - r}{\sigma} \right)^2 + (\nu_u)^2 \right] du \right. \\ &\quad \left. - \int_0^t \frac{\mu - r}{\sigma} dz_{1,u} - \int_0^t \nu_u dz_{2,u} \right\} \end{aligned} \quad (4.21)$$

is the *state-price density* (SPD).

Let us denote by  $H$  the logarithm of the SPD  $\xi^{(\nu)}$  as defined in (4.21). i.e.,  $H = \ln \xi^{(\nu)}$ . Since in equilibrium, the state price density must be path-independent, we can specify  $H$  as a function of  $S$ ,  $r$ , and  $t$  only. Namely,  $H \equiv H(S, r, t)$ . Comparing  $dz_1$  and  $dz_2$  terms of  $dH$  with those of  $d \ln(\xi^{(\nu)})$  produces

$$\begin{aligned}\sigma(S, r, t)S(t)H_S(S, r, t) + \rho\beta(r, t)H_r(S, r, t) &= -\frac{\mu(S, r, t) - r(t)}{\sigma(S, r, t)}, \\ \beta(r, t)\sqrt{1 - \rho^2}H_r(S, r, t) &= -\nu(S, r, t).\end{aligned}$$

It follows that

$$S(t)H_S(S, r, t) = -\frac{\mu(S, r, t) - r(t)}{\sigma^2(S, r, t)} + \frac{\rho\nu(S, r, t)}{\sigma(S, r, t)\sqrt{1 - \rho^2}} \equiv -f(S, r, t), \quad (4.22)$$

and

$$H_r(S, r, t) = -\frac{\nu(S, r, t)}{\beta(r, t)\sqrt{1 - \rho^2}} \equiv -g(S, r, t). \quad (4.23)$$

The following theorem provides the equilibrium condition that the parameters of the stock index, the interest rate processes and the factor risk premium must satisfy in order to be consistent with equilibrium. The characterisation is obtained by imposing (1) the stock price process is an equilibrium price process in the sense that the demand and supply of the stock are equal ; (2) the factor risk premium function  $\nu$  determines the equilibrium shadow prices for time-and-state contingent claims.

**Theorem 4.1** *The necessary and sufficient conditions for  $(S, r)$  defined in (4.1) and (4.2) to be consistent with an equilibrium are that there exists a factor risk*

premium function  $\nu$  such that  $(f, g)$  as defined in (4.22) and (4.23) satisfies (C1)-(C3):

(C1) the functions  $f(S, r, t)$  and  $g(S, r, t)$  defined in (4.22) and (4.23) must satisfy

$$f_r = Sg_S; \quad (4.24)$$

(C2) given all coefficients and  $g$  defined in (4.23), a non-linear second-order PDE in terms of  $f$  must be satisfied:

$$\mathcal{L}f + f_t + S\sigma\sigma_S(Sf_S + f^2 - f) + \beta(\rho\sigma_S S - \beta g)f_r - \rho\sigma\beta Sgf_S = 0, \quad (4.25)$$

where

$$\mathcal{L}f = \mu Sf_S + \alpha f_r + \frac{1}{2}\sigma^2 S^2 f_{SS} + \frac{1}{2}\beta^2 f_{rr} + \rho\sigma\beta S f_{Sr};$$

(C3) given all coefficients and  $f$  defined in (4.22), a non-linear second-order PDE in terms of  $g$  must be satisfied:

$$\begin{aligned} \mathcal{L}g + g_t + \rho[-\sigma\beta g + (\sigma\beta)_r]Sg_S + \beta(\beta_r - \beta g)g_r + (\alpha_r - \beta\beta_r g)g \\ + (f - 1) + \sigma\sigma_r(Sf_S + f^2 - f) = 0, \end{aligned} \quad (4.26)$$

where

$$\mathcal{L}g = \mu Sg_S + \alpha g_r + \frac{1}{2}\sigma^2 S^2 g_{SS} + \frac{1}{2}\beta^2 g_{rr} + \rho\sigma\beta S g_{Sr};$$

**Proof:** Since  $H_{Sr} = H_{rS}$ , we differentiate (4.22) with respect to  $r$  and (4.23) with respect to  $S$  to obtain  $SH_{Sr} = -f_r$  and  $H_{rS} = -g_S$ . Hence proved (C1).

To prove (C2), we first compare the  $dt$  term of  $dH$  with that of  $d\ln(\xi^{(\nu)})$  to obtain an equality expressed in shorthand as

$$\begin{aligned} \mu SH_S + \alpha H_r + H_t + \frac{1}{2}\sigma^2 S^2 H_{SS} + \frac{1}{2}\beta^2 H_{rr} + S\sigma\beta\rho H_{Sr} \\ = -r - \frac{1}{2} \left[ \left( \frac{\mu - r}{\sigma} \right)^2 + \nu^2 \right]. \end{aligned} \quad (4.27)$$



Next, we differentiate (4.22) with respect to  $S$  and  $r$ , respectively. and (4.23) with respect to  $r$  to obtain  $S^2 H_{SS} = f - S f_S$ ,  $S H_{Sr} = -f_r$  and  $H_{rr} = -g_r$ . Substituting those into (4.27) together with (4.22) and (4.23) and realising that  $\mu - r = \sigma^2 f + \rho \sigma \beta g$  yield

$$\begin{aligned} \frac{1}{2} \sigma^2 (-S f_S + f) - \rho \sigma \beta f_r - \frac{1}{2} \beta^2 g_r - \frac{1}{2} \sigma^2 f^2 - r f - \alpha g + H_t \\ = -\frac{1}{2} \beta^2 g^2 - r. \end{aligned} \quad (4.28)$$

Equation (4.28) can be further differentiated with respect to  $S$  to obtain an expression with  $H_{tS}$  and  $g_{rS}$  appearing. Eventually, (C2) can be proved by using the facts that  $S g_{rS} = f_{rr}$  and  $S H_{St} = -f_t$ . (C3) can also be proved by following a similar approach, starting from differentiating (4.28) with respect to  $r$ . The sufficiency part of the proof can be referred to the Appendix in He [57].  $\square$

### 4.3.3 The Representative Agent's Utility Function

By inspecting (4.22) and (4.23), one gets an indication about how one might be able to back out the representative agent's marginal utility function that supports the equilibrium price system. The following theorem delivers the result.

**Theorem 4.2** *Assume in equilibrium the representative agent optimally holds the stock index (or equivalently the market portfolio) at all times. Given the asset price dynamics (4.1) and the interest rate process (4.2), the marginal utility function which supports the system is given by*

$$U_S = \exp \left\{ - \int_{r_0}^r g(S_0, y, T) dy \right\} \exp \left\{ \int_{S_0}^S - \frac{f(x, r, T)}{x} dx \right\}. \quad (4.29)$$

**Proof:** See He [57], pp.15-16.  $\square$

Note that in (4.29), the first exponent is a function of  $r$  only. Therefore, if  $f$  is a function of  $S$  and  $T$  only, then the utility is separable in wealth and the interest rate  $r$ .

In principle, one can use Theorem 4.1 to derive systematically all of the admissible factor risk premium functions  $\nu$  that are consistent with our underlying equilibrium. However, without further assumptions, it is generally extremely difficult to solve these complicated differential equations. Nevertheless, from (4.29), we can observe some important special cases. First, the utility function is state-independent when  $g = 0$ . Second, when  $f$  is constant and  $g = 0$ , the representative agent exhibits a constant relative risk aversion (CRRA) utility. Finally, if we assume the factor risk premium  $\nu$  is a function of  $r$  and  $t$  only, i.e.  $\nu \equiv \nu(r, t)$ , then  $g \equiv g(r, t)$  since  $\nu_S = 0$  from (4.23) and  $f \equiv f(S, t)$  since  $g_S = 0$  from (4.24). Therefore, from (4.29), the utility function must be separable in  $W$  and  $r$ .

One further simplification is to assume that  $\mu$ ,  $\sigma$ ,  $\alpha$ , and  $\beta$  are all time-homogeneous and  $\nu$  is a function of  $r$  only. Therefore, from (4.22) and (4.23),  $g$  and  $f$  will also be time-homogeneous, i.e.,  $g = g(r)$  and  $f = f(S)$ . (4.25) then reduces to

$$2rf_S + \frac{\partial}{\partial S} [\sigma^2(f^2 + Sf_S - f)] = 0,$$

or equivalently,

$$2rf + \sigma^2(f^2 + Sf_S - f) = m(r); \quad (4.30)$$

Similarly, (4.26) reduces to

$$\frac{1}{2}\beta^2 g_r + \alpha g - \frac{1}{2}\beta^2 g^2 + \frac{1}{2}\sigma^2(f^2 + S f_S - f) + r(f - 1) = K. \quad (4.31)$$

where  $K$  is an arbitrary constant.

(4.30) and (4.31) provide a systematic way to find  $f$  and  $g$  which determine the utility function. For example, if  $\sigma$  is further specified as a function of  $r$  only, then a constant  $f$  is a solution to (4.30). Therefore, (4.31) becomes a first-order non-linear ODE which is supported by a state-dependent CRRA type utility function scaled by a constant  $c$ :

$$U = c \exp \left\{ - \int_{r_0}^r g(S_0, y, T) dy \right\} \cdot \frac{S_T^{1-f}}{1-f}. \quad (4.32)$$

### 4.3.4 Equilibrium Equity Derivatives Pricing

As long as the equilibrium factor risk premium function  $\nu$  is known, one can price any European contingent claims written on the assets. It is not uncommon in the option pricing literature to specify  $\nu$  exogenously. Whether the specified processes for  $S(t)$ ,  $r(t)$  and  $\nu(t)$  are consistent with an equilibrium depends upon the definition of equilibrium we employ. Recall that the kind of equilibrium we assumed in this analysis is such that under equilibrium, the representative agent must optimally hold the market at any time prior to the horizon date. It is also assumed that there will be no dividend and intermediate consumption. This is obviously a more restricted form than a dynamically competitive equilibrium market.

In our context, denote by  $C$  the value of a European call option written on the



stock index with exercise price  $X$ . The value  $C$  must satisfy the following PDE:

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + \rho\sigma\beta S C_{Sr} + \frac{1}{2}\beta^2 C_{rr} + r S C_S + (\alpha - \Lambda)C_r - rC = 0, \quad (4.33)$$

with the boundary condition  $C(S, r, T) = (S - X)^+$ .

## 4.4 Vasicek's One-Factor Short Rate Model

In this section, we will apply the results from previous sections to a special case where the short rate follows a mean-reverting (Ornstein-Uhlenbeck) process. Specifically, we assume that the stock index offers a constant risk premium and the interest rate follows Vasicek's [103] short rate model, i.e.

$$\frac{dS_t}{S_t} = (r_t + \lambda)dt + \sigma dz_{1,t}, \quad (4.34)$$

$$dr_t = \kappa(\theta - r_t)dt + b \left( \rho dz_{1,t} + \sqrt{1 - \rho^2} \right) dz_{2,t}, \quad (4.35)$$

where  $\lambda$ ,  $\sigma$ ,  $\kappa$ ,  $\theta$ ,  $b$ , and  $\rho$  are constant.

To find the supporting economy, we first obtain the functions  $f$  and  $g$  from (4.22) and (4.23),

$$SH_S = -\frac{\lambda}{\sigma^2} + \frac{\rho\nu}{\sigma\sqrt{1 - \rho^2}} \equiv -f(S, r, t), \quad (4.36)$$

$$H_r = -\frac{\nu}{b\sqrt{1 - \rho^2}} \equiv -g(S, r, t). \quad (4.37)$$

The next task is to find feasible factor risk premium functions  $\nu$  which are consistent with equilibrium. In order to be able to solve the problem explicitly, we confine ourself to the time-homogeneous case where  $\nu$  is a function of  $r$  only. The following proposition states the equilibrium price system and the supporting utility function.

**Proposition 4.1** *Assume the processes (4.34) and (4.35) and the factor risk premium function  $\nu$  is a function of  $r$  only. The necessary and sufficient conditions for the price system to be consistent with the equilibrium are that*

$$\lambda = \gamma\sigma^2 + (\gamma - 1)\frac{\rho b\sigma}{\kappa}, \quad (4.38)$$

$$\nu = b\sqrt{1 - \rho^2} \left( \frac{\gamma - 1}{\kappa} \right), \quad (4.39)$$

$$\frac{\rho b}{\kappa\sigma} < \frac{\gamma}{1 - \gamma}, \text{ when } 0 < \gamma < 1, \quad (4.40)$$

$$\frac{\rho b}{\kappa\sigma} > \frac{\gamma}{1 - \gamma}, \text{ when } \gamma > 1. \quad (4.41)$$

*The supporting utility function of the representative agent is*

$$U = \exp \left\{ \frac{1 - \gamma}{\kappa} (r_T - r_0) \right\} \frac{W_T^{1 - \gamma}}{1 - \gamma}, \quad (4.42)$$

for  $\gamma > 0, \gamma \neq 1$ , and is  $U = \ln W_T$ , for  $\gamma = 1$ .  $r_0$  and  $r_T$  are interest rates at time 0 and time  $T$ , respectively.

**Proof:** From previous analysis,  $g$  will be a function of  $r$  only and  $f$  will be a function of  $S$  only. By inspection of (4.36) and (4.37), it turns out that  $\nu$  is constant, so are  $f$  and  $g$ . Condition (4.30) is apparently satisfied. Functions  $f$  and  $g$  can be related by (4.36) and (4.37). When we define  $\gamma \equiv f$ , it follows that

$$\gamma = \frac{\lambda}{\sigma^2} - \frac{\rho b}{\sigma} g. \quad (4.43)$$

From (4.31), we obtain

$$\kappa(\theta - r)g - \frac{1}{2}b^2g^2 + \frac{1}{2}\sigma^2(\gamma^2 - \gamma) + r(\gamma - 1) = K.$$

By letting the coefficient of  $r$  be zero, we obtain  $g = \frac{\gamma - 1}{\kappa}$ . Substituting  $g$  into (4.43) and rearranging the equation yield (4.38). Equation (4.39) also follows



immediately from (4.37). The sufficient conditions (4.40) and (4.41) are given to ensure a positive market risk premium  $\lambda$  for  $\gamma > 0$ ,  $\gamma \neq 1$ . When  $\gamma = 1$ , it is obvious that  $\lambda = \sigma^2 > 0$ . The supporting utility function (4.42) is a direct result from (4.32).  $\square$

Note that in this example, the utility function which supports our kind of equilibrium is, in general, state-dependent unless the investor's coefficient of relative risk aversion is equal to one. Differentiating (4.42) with respect to  $r_T$  yields

$$U_{r_T} = \frac{1-\gamma}{\kappa} U. \quad (4.44)$$

It can be shown that  $U_{r_T}$  is positive if  $\gamma > 0$  and  $\gamma \neq 1$ , and is zero if  $\gamma = 1$ . In other words, the representative investor prefers higher interest rate to lower interest rate. However, the marginal utility gain with respect to  $r$  will be increasing (decreasing) if  $0 < \gamma < 1$  ( $\gamma > 1$ ).

## 4.5 Explicit Solutions to Non-Representative Agent's Optimal Asset Allocation

Following the analysis in the previous section, we now solve explicitly a non-representative agent's portfolio choice problem. The agent is assumed to have the same horizon as the economy. It is conventional in the related literature to consider a state-independent CRRA utility. Therefore, we first solve the problem for an agent with this type of utility. However, the interesting result that the economy we described is supported by a representative agent with a state-dependent CRRA type utility enables us to consider a non-representative agent with the same form

of utility as the representative agent, i.e. (4.42), but with risk aversion  $\eta$  rather than  $\gamma$ . We shall refer to the agent with the first type of utility the Type 1 agent, and the one with the second type the Type 2 agent. Since we assume the market is complete, we can apply the martingale technique to obtain explicit solutions for both cases.

#### 4.5.1 Non-Representative Agent with Terminal Utility Independent of Interest Rate

Consider a non-representative agent with a CRRA utility:

$$U(W_T, T) = \frac{W_T^{1-\eta}}{1-\eta}, \quad (4.45)$$

where  $\eta > 0$ , and  $\eta \neq 1$ . When  $\eta = 1$ , the utility is of a logarithmic form  $U(W_T, T) = \ln W_T$ . We shall call the agent with this type of utility the Type 1 agent.

Since the market is complete, from the first order condition, we can rearrange to obtain

$$W_T = l^{-\frac{1}{\eta}} \xi_T^{-\frac{1}{\eta}}, \quad (4.46)$$

where  $l$  is the Lagrange multiplier and  $\xi$  is the state-price density as defined in (4.21). Note that for notational simplicity, we have suppressed the superscript  $\nu$ . Substituting (4.46) into the budget constraint,  $\xi_t \cdot W_t = \mathbb{E}_t[\xi_T \cdot W_T]$  and rearranging the equation yield

$$W_t = l^{-\frac{1}{\eta}} \xi_t^{-\frac{1}{\eta}} \mathbb{E}_t \left[ \left( \frac{\xi_T}{\xi_t} \right)^{1-\frac{1}{\eta}} \right]. \quad (4.47)$$



From (4.46) and (4.47), we obtain

$$W_T = W_t \left( \frac{\xi_T}{\xi_t} \right)^{-\frac{1}{\eta}} \left\{ \mathbb{E}_t \left[ \left( \frac{\xi_T}{\xi_t} \right)^{1-\frac{1}{\eta}} \right] \right\}^{-1}. \quad (4.48)$$

Therefore, the utility function becomes

$$U(W_T, T) = \frac{W_t^{1-\eta}}{1-\eta} \left( \frac{\xi_T}{\xi_t} \right)^{1-\frac{1}{\eta}} \cdot \left\{ \mathbb{E}_t \left[ \left( \frac{\xi_T}{\xi_t} \right)^{1-\frac{1}{\eta}} \right] \right\}^{\eta-1}. \quad (4.49)$$

By applying the techniques of term-structure models, it follows that the indirect utility function is of the form

$$J(W_t, r_t, t) = \mathbb{E}_t[U(W_T, T)] = \frac{(W_t/P_t)^{1-\eta}}{1-\eta} e^{\frac{1-\eta}{2\eta}v(t,T)}, \quad (4.50)$$

where  $P_t \equiv P(r, t; T)$  is the time- $t$  price of the  $T$ -bond, and <sup>2</sup>

$$\begin{aligned} v(t, T) &= (\mathcal{A} + 2\mathcal{B} + \mathcal{C})\tau - (2\mathcal{B} + \mathcal{C})D(t, T) - \frac{\kappa}{2}\mathcal{C}(D(t, T))^2, \\ \tau &= T - t, \\ D(t, T) &= \frac{1 - e^{-\kappa\tau}}{\kappa}, \\ \mathcal{A} &= \left( \frac{\lambda}{\sigma} \right)^2 + \nu^2, \\ \mathcal{B} &= \frac{b}{\kappa} \left[ \rho \left( \frac{\lambda}{\sigma} \right) + \sqrt{1 - \rho^2} \nu \right], \\ \mathcal{C} &= \frac{b^2}{\kappa^2}. \end{aligned}$$

As a result, we can derive the partial derivatives,  $J_W$ ,  $J_{WW}$ ,  $J_{Wr}$  and  $J_{WS}$ , where  $J_{WS} = 0$ . Substituting them into (4.13) and (4.14) and applying (4.38) yield the optimal choice. Let  $\Sigma$  denote the  $2 \times 2$  variance-covariance matrix of the

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<sup>2</sup> $W_t/P_t$  can be interpreted as the forward price of a forward contract written on the investor's time- $T$  wealth in absolute term. Hence, the investor is indifferent between allocating across assets dynamically and selling short the forward contract which guarantees the payoff of  $W_T$  at  $T$ .

returns on the stock index and the  $T$ -bond. Also, let  $\underline{\pi}$  denote the  $2 \times 1$  vector of risk premia. It is a well-known result that, in this case, the optimal holdings on these two assets are given by

$$\begin{pmatrix} \phi_{S,1}^* \\ \phi_{P,1}^* \end{pmatrix} = \frac{1}{\eta} \Sigma^{-1} \underline{\pi} + \left(1 - \frac{1}{\eta}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.51)$$

As in Merton [85], (4.51) shows that the optimal portfolio is a linear combination of the myopic policy  $\Sigma^{-1} \underline{\pi}$  which is instantaneously mean-variance efficient and a hedge portfolio which contains only the  $T$ -bond. The weights on these two portfolios depend on the risk aversion parameter  $\eta$ .

Given the result from Proposition 4.1, the optimal proportions of wealth invested in the stock index, the  $T$ -bond for the Type 1 agent can be expressed as

$$\begin{pmatrix} \phi_{S,1}^* \\ \phi_{P,1}^* \end{pmatrix} = \begin{pmatrix} \frac{\gamma}{\eta} \\ \frac{1}{1-e^{-\kappa\tau}} \left(\frac{1-\gamma}{\eta}\right) \end{pmatrix} + \left(1 - \frac{1}{\eta}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.52)$$

The proportion of wealth held in the bank account is therefore

$$\phi_{R,1}^* = \frac{e^{-\kappa\tau}}{1 - e^{-\kappa\tau}} \left(\frac{\gamma - 1}{\eta}\right).$$

Equations (4.51) and (4.52) indicate that the optimal stock investment is a myopic policy and, in fact, a constant policy. This is not surprising since we have assumed a constant risk premium for the stock index. However, the demand function for the  $T$ -bond displays a time-varying behaviour which reflects the time-varying opportunity set. The demand on the  $T$ -bond will increase as the time to maturity lengthens, until in the limit case as  $\tau \rightarrow \infty$ ,  $\phi_P^* \rightarrow 1 - \gamma/\eta$ .



### 4.5.2 Non-Representative Agent with Terminal Utility of Same Form as Representative Agent but Different RRA Coefficient

Now let us consider a non-representative agent with utility function of the same form as representative agent but with different risk aversion  $\eta$ :

$$U = \exp \left\{ \frac{1-\eta}{\kappa} (r_T - r_0) \right\} \frac{W_T^{1-\eta}}{1-\eta}, \quad (4.53)$$

where  $\eta \neq 1$ ,  $U = \ln W_T$  when  $\eta = 1$ . We shall call the agent with this type of utility the Type 2 agent.

Using the same approach as for the Type 1 agent, we can re-derive

$$W_T = l^{-\frac{1}{\eta}} \xi_T^{-\frac{1}{\eta}} e^{\frac{1-\eta}{\kappa\eta} (r_T - r_0)} \quad (4.54)$$

and also

$$W_t = l^{-\frac{1}{\eta}} \xi_t^{-\frac{1}{\eta}} \mathbb{E}_t \left[ \left( \frac{\xi_T}{\xi_t} \right)^{1-\frac{1}{\eta}} \cdot e^{\frac{1-\eta}{\kappa\eta} (r_T - r_0)} \right]. \quad (4.55)$$

From (4.54) and (4.55),

$$W_T = \frac{W_t \left( \frac{\xi_T}{\xi_t} \right)^{-\frac{1}{\eta}} \cdot e^{\frac{1-\eta}{\kappa\eta} (r_T - r_0)}}{\mathbb{E}_t \left[ \left( \frac{\xi_T}{\xi_t} \right)^{1-\frac{1}{\eta}} \cdot e^{\frac{1-\eta}{\kappa\eta} (r_T - r_0)} \right]} \quad (4.56)$$

and the utility function becomes

$$\begin{aligned} U(W, r, T) &= e^{\frac{1-\eta}{\kappa\eta} (r_T - r_0)} \frac{W_t^{1-\eta}}{1-\eta} \left( \frac{\xi_T}{\xi_t} \right)^{1-\frac{1}{\eta}} \\ &\quad \cdot \left\{ \mathbb{E}_t \left[ \left( \frac{\xi_T}{\xi_t} \right)^{1-\frac{1}{\eta}} e^{\frac{1-\eta}{\kappa\eta} (r_T - r_0)} \right] \right\}^{\eta-1}. \end{aligned} \quad (4.57)$$

Hence, the indirect utility function is given by

$$J(W, S, r, t) = \mathbb{E}_t[U(W, r, T)] = \frac{(W_t/P_t)^{1-\eta}}{1-\eta} e^{(1-\eta)v_1(r,t;T)} e^{\frac{1-\eta}{2\eta}v_2(t,T)} e^{(1-\eta)v_3(t,T)} \quad (4.58)$$

where

$$\begin{aligned} v_1(r, t; T) &= \frac{1}{\kappa} (e^{-\kappa\tau} r_t - r_0), \\ v_2(t, T) &= \mathcal{A} + 2\mathcal{B} + \mathcal{C}, \\ v_3(t, T) &= \left( \theta - \mathcal{B} - \frac{1}{2}\mathcal{C} \right) D(t, T) - \frac{\kappa}{4}\mathcal{C} (D(t, T))^2. \end{aligned}$$

and  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are as defined earlier.

Deriving the partial derivatives,  $J_W$ ,  $J_{WW}$ ,  $J_{Wr}$  and  $J_{WS}$ , where  $J_{WS} = 0$  and substituting them into (4.13) and (4.14) yield the optimal choice for the Type 2 agent. In matrix form, the optimal holdings on the stock and the  $T$ -bond are given by

$$\begin{pmatrix} \phi_{S,2}^* \\ \phi_{P,2}^* \end{pmatrix} = \frac{1}{\eta} \Sigma^{-1} \underline{\pi} + \left( 1 - \frac{1}{\eta} \right) \begin{pmatrix} 0 \\ \frac{1}{1-e^{-\kappa\tau}} \end{pmatrix}. \quad (4.59)$$

As in the case of Type 1 agent, the hedge portfolio contains only the  $T$ -bond. However, being different from the Type 1 agent, the hedge demand is now time-varying instead of constant. Again, we can express the optimal portfolio in terms of  $\gamma$  and  $\eta$  as

$$\begin{pmatrix} \phi_{S,2}^* \\ \phi_{P,2}^* \end{pmatrix} = \begin{pmatrix} \frac{\gamma}{\eta} \\ \frac{1}{1-e^{-\kappa\tau}} \left( \frac{1-\gamma}{\eta} \right) \end{pmatrix} + \left( 1 - \frac{1}{\eta} \right) \begin{pmatrix} 0 \\ \frac{1}{1-e^{-\kappa\tau}} \end{pmatrix}. \quad (4.60)$$

The proportion of wealth held in the bank account is, therefore, given by

$$\phi_{R,2}^* = \frac{e^{-\kappa\tau}}{1 - e^{-\kappa\tau}} \left( \frac{\gamma}{\eta} - 1 \right).$$

Note that as found in the Type 1 agent, Type 2 agent behaves in a similar way with regard to the risk aversion. The optimal holding for the stock decreases and the optimal holding for the  $T$ -bond increases when the agent becomes more and more risk averse. In the limit case, when the agent has infinite risk aversion (i.e.  $\eta \rightarrow \infty$ ),  $\phi_{S,2}^* = 0$  and  $\phi_{P,2}^* = 1/(1 - e^{-\kappa\tau})$ .



## 4.6 Numerical Illustrations

In this section, we illustrate our portfolio selection models derived in Section 4.5 by applying realistic parameter values and evaluating the relevant functions numerically. Our main interests are (1) to see how these two types of agents assumed in the analysis behave in their optimal allocation strategies and (2) to examine how they behave in the limit case as the time to horizon approaches infinity.

The parameters driving the stock index and interest rates processes are chosen as follows. For the stock index dynamics, the risk premium is  $\lambda = 0.6$  and the volatility is  $\sigma = 0.2$ . For the interest rate dynamics,  $b = 0.02$  and  $\kappa$  varies according to  $\gamma$ . The correlation between  $dz_1$  and  $dz_2$  is  $\rho = -0.25$ .

Figure 4.1 shows the optimal portfolio choices against the ratio of  $\gamma$  to  $\eta$  for both types of non-representative agents. The representative agent's relative risk aversion is  $\gamma = 2$  and the time to maturity is 10 years. Panel I shows the optimal portfolio choice of an agent with a state-independent CRRA utility. Panel II shows the optimal portfolio choice of an agent with a state-dependent utility which is of the same form as the representative agent but possibly different risk aversion. The solid lines represent the holdings in the stock,  $\phi_S$ , the dashed lines represent the holdings in the  $T$ -bond,  $\phi_P$ , and the dash-dot lines represent the holdings in the bank account,  $\phi_R$ . In both cases, the optimal holding for the stock decreases and the optimal holding for the  $T$ -bond increases when the agent is more risk averse.

Detailed results are given in Tables 4.1-4.3. Table 4.1 tabulates the optimal holding in the stock index. From (4.52) and (4.60), we know that  $\phi_{S,1} = \phi_{S,2}$

and it decreases as the agent becomes more risk averse. Tables 4.2 and 4.3 give the optimal holding in the  $T$ -bond for the Type 1 agent and the Type 2 agent, respectively. It can be seen that the optimal holding in the  $T$ -bond increases as the agent becomes more risk averse. Therefore, one can conclude that the bond-stock ratio should be increasing in the risk aversion. This is consistent with the conventional wisdom and the common recommendations given by financial analysts.

As analysed in the model, the speculative demands for the  $T$ -bond are the same for both utility assumptions. Hence, Figure 4.2 shows only the hedge demand against the ratio of  $\gamma$  to  $\eta$  for both types of non-representative agents. Specifically, it plots the demand function for the  $T$ -bond since the hedge portfolio contains only the  $T$ -bond. The representative agent's relative risk aversion is  $\gamma = 2$  and the time to maturity is 10 years.

Finally, the optimal holding in the  $T$ -bond for the limit case as  $\tau \rightarrow \infty$  is given by Table 4.4. Our model shows that it is the same for both types of agents. Figure 4.3 shows the optimal holdings in the stock index and the  $T$ -bond across different horizons. Panel I shows the result for the case where  $\gamma = 3$  and  $\eta = 2$ . Panel II shows the result for the case where  $\gamma = 2$  and  $\eta = 5$ . While the holdings in the  $T$  bond converge as  $\tau$  increases, the behaviour could be quite different. In particular, when  $\eta > \gamma$ , the hedge demand of a Type 2 agent will dominate so that she will hold more and more bond over time. As for the Type 1 agent, the speculative demand will dominate the (constant) hedge demand so that she will hold less and less bond over time regardless of the relative level of risk aversion.



Note that although the qualitative properties proposed by our models are quite encouraging, there appears to be a singularity when  $\tau = 0$ . That is, from Figure 4.3, non-representative agents either purchase or sell short infinite amounts of bonds when approaching the horizon date - this is as a result of the form of demand functions (4.52) and (4.60). It can be argued that in a discrete time setting, this should not pose a serious problem. However, it would be nice if we can have a model that prevents the singular behaviour even in the continuous time. We conjecture that introducing intermediate consumption might be able to correct this problem.

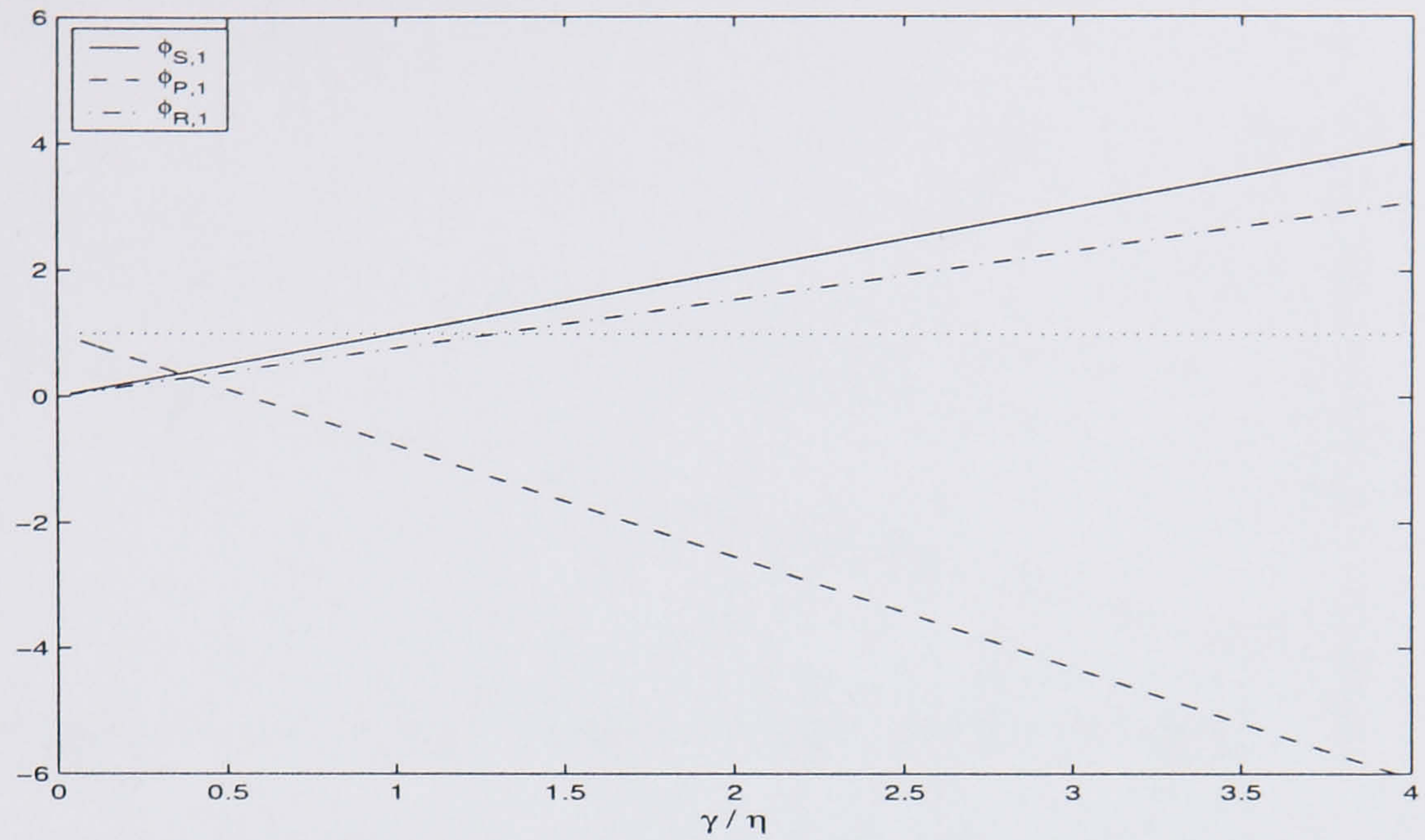
## 4.7 Chapter Summary and Conclusions

In this chapter, the dynamic asset allocation problem under stochastic interest rates was investigated. We first characterised the equilibrium conditions which the asset price dynamics must satisfy and then applied the results to the Vasicek's one-factor short rate model. The dynamic asset allocation problem for a non-representative agent was then solved explicitly using the dual approach.

We assumed two types of utility for the non-representative agent. In both cases, the optimal holding for the stock index is constant (also myopic) and the hedge portfolio contains only the  $T$ -bond. The main difference, however, is in the hedge demand for the  $T$ -bond. When the agent has the utility of the same form as representative agent but with different risk aversion, the hedge demand for the  $T$ -bond is time-varying instead of being constant as in the CRRA case. We showed that, given reasonable levels of risk aversion, the agent should purchase more and



Panel I



Panel II

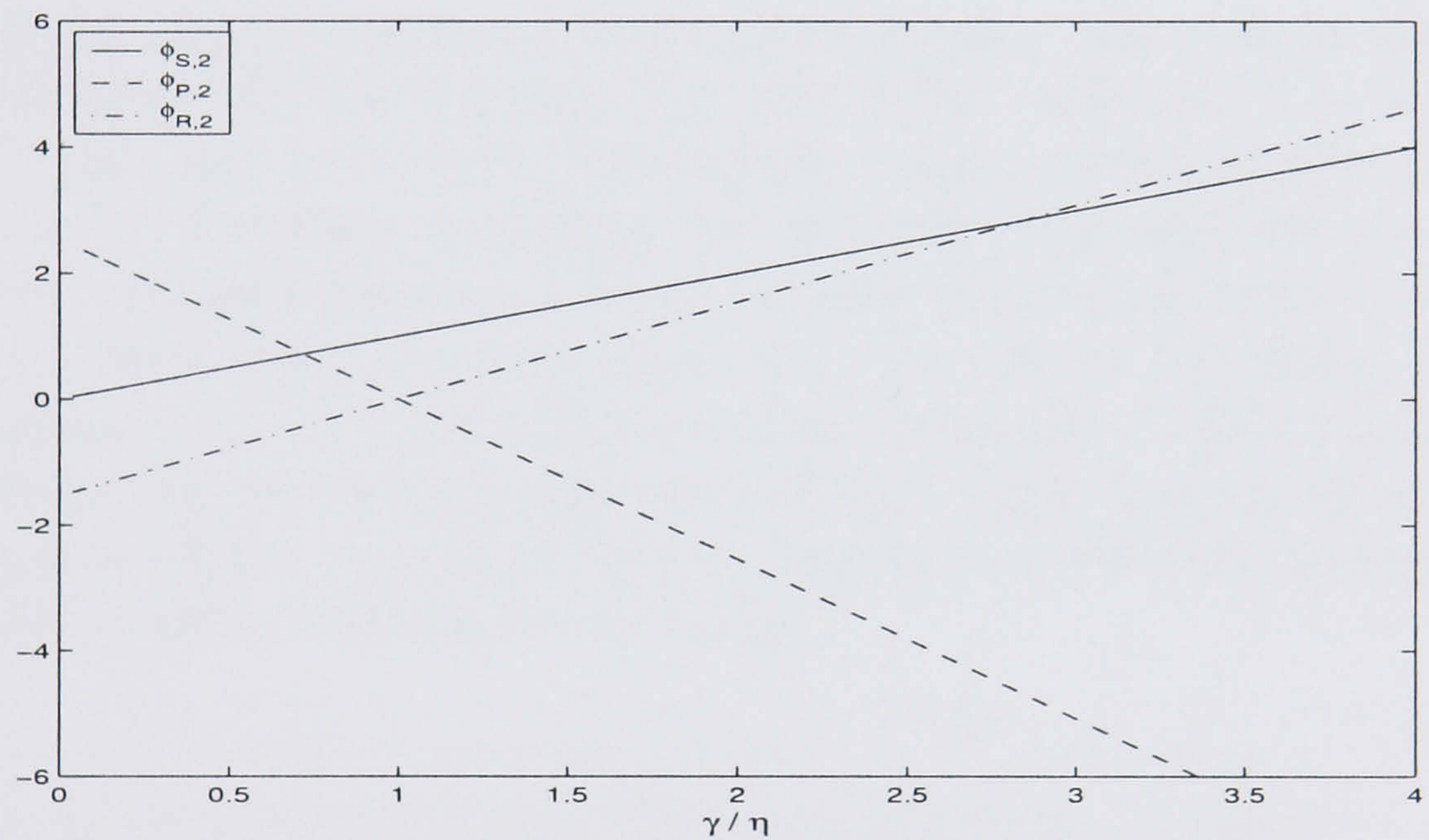


Figure 4.1: Optimal portfolio choices of non-representative agents. These figures show the agent's optimal holdings in the stock index, the  $T$ -bond and the bank account. The parameter values are  $\lambda = 0.06$ ,  $\sigma = 0.2$ ,  $b = 0.02$ , and  $\rho = -0.25$ . The representative agent's relative risk aversion is  $\gamma = 2$ . The time to maturity is 10 years. Panel I shows the optimal portfolio choice of a Type 1 agent who has a state-independent CRRA utility. Panel II shows the optimal portfolio choice of a Type 2 agent who has a state-dependent CRRA type utility of the same form as the representative agent but with possibly different risk aversion. The solid lines represent the holdings in the stock,  $\phi_S$ , the dashed lines represent the holdings in the  $T$ -bond,  $\phi_P$ , and the dash-dot lines represent the holdings in the bank account,  $\phi_R$ .



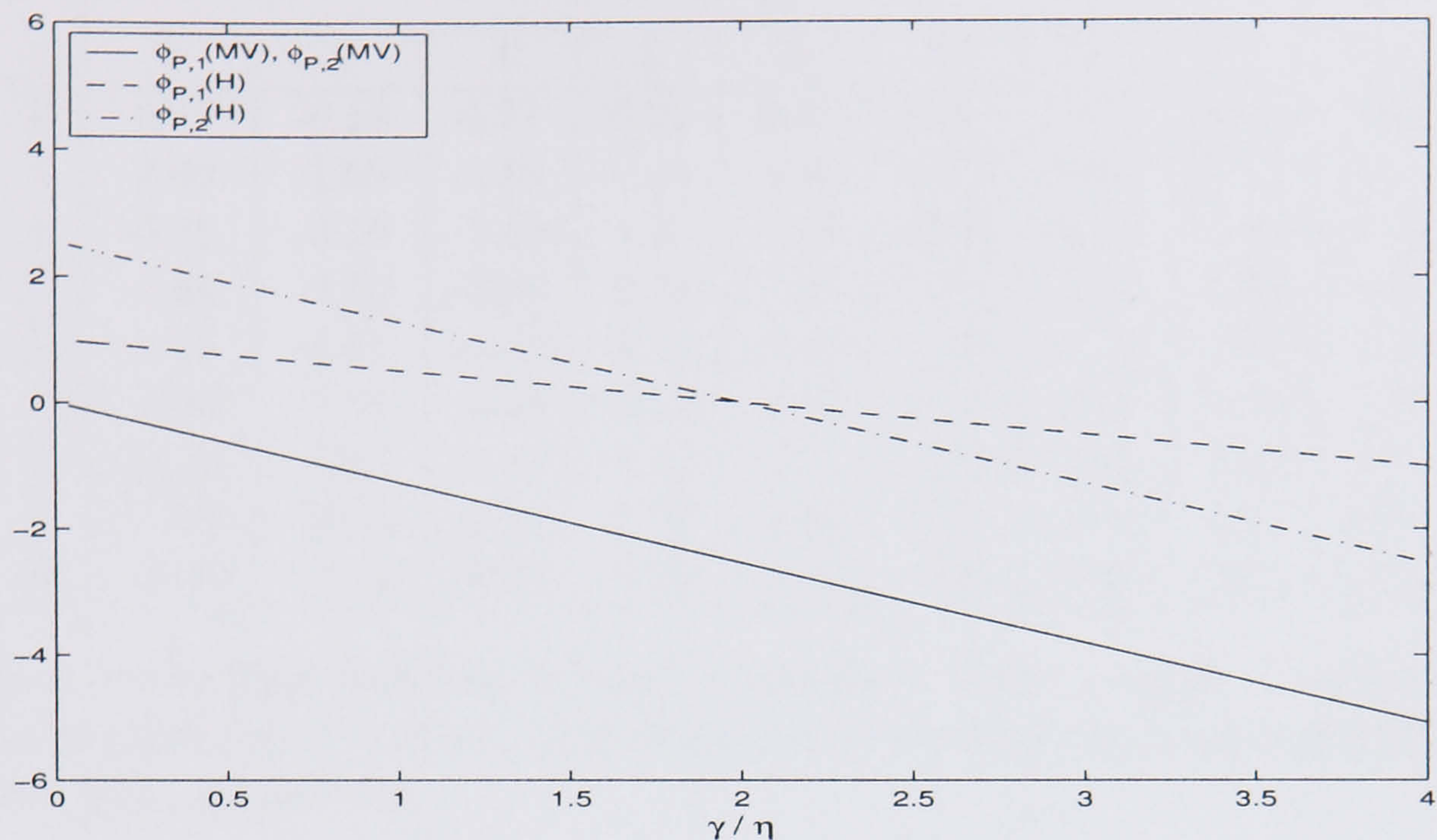


Figure 4.2: Hedge demand of non-representative agents. These figure plots the demand function for the  $T$ -bond. The parameter values are  $\lambda = 0.06$ ,  $\sigma = 0.2$ ,  $b = 0.02$ , and  $\rho = -0.25$ . The representative agent's relative risk aversion is  $\gamma = 2$ . The time to maturity is 10 years. The solid line shows the demand for the instantaneously mean-variance efficient portfolio which is the same for both types of utility functions. The dashed line shows the hedge demand for a Type 1 agent who has a state-independent CRRA utility and the dash-dot line shows the hedge demand for a Type 2 agent who has a state-dependent CRRA type utility of the same form as the representative agent but with possibly different risk aversion.

$\gamma$	$\eta$								
	2	3	4	5	6	7	8	9	10
2	1.00	0.67	0.50	0.40	0.33	0.29	0.25	0.22	0.20
3	1.50	1.00	0.75	0.60	0.50	0.43	0.38	0.33	0.30
4	2.00	1.33	1.00	0.80	0.67	0.57	0.50	0.44	0.40
5	2.50	1.67	1.25	1.00	0.83	0.71	0.63	0.56	0.50
6	3.00	2.00	1.50	1.20	1.00	0.86	0.75	0.67	0.60
7	3.50	2.33	1.75	1.40	1.17	1.00	0.88	0.78	0.70
8	4.00	2.67	2.00	1.60	1.33	1.14	1.00	0.89	0.80
9	4.50	3.00	2.25	1.80	1.50	1.29	1.13	1.00	0.90
10	5.00	3.33	2.50	2.00	1.67	1.43	1.25	1.11	1.00

Table 4.1: Optimal holding in the stock index.  $\gamma$  denotes the representative agent's RRA coefficient and  $\eta$  denotes the non-representative agent's RRA coefficient.



$\gamma$	$\eta$								
	2	3	4	5	6	7	8	9	10
2	-0.77	-0.18	0.11	0.29	0.41	0.49	0.56	0.61	0.65
3	-3.03	-1.69	-1.01	-0.61	-0.34	-0.15	-0.01	0.10	0.19
4	-5.29	-3.19	-2.14	-1.51	-1.10	-0.80	-0.57	-0.40	-0.26
5	-7.55	-4.70	-3.27	-2.42	-1.85	-1.44	-1.14	-0.90	-0.71
6	-9.81	-6.21	-4.40	-3.32	-2.60	-2.09	-1.70	-1.40	-1.16
7	-12.07	-7.71	-5.53	-4.23	-3.36	-2.73	-2.27	-1.90	-1.61
8	-14.33	-9.22	-6.66	-5.13	-4.11	-3.38	-2.83	-2.41	-2.07
9	-16.59	-10.73	-7.79	-6.04	-4.86	-4.03	-3.40	-2.91	-2.52
10	-18.85	-12.23	-8.92	-6.94	-5.62	-4.67	-3.96	-3.41	-2.97

Table 4.2: Optimal holding in the  $T$ -bond for a Type 1 agent.  $\gamma$  denotes the representative agent’s RRA coefficient and  $\eta$  denotes the non-representative agent’s RRA coefficient.

more bond over time due to the increasing hedge demand. They act as such to hedge the possible loss of utility should the interest rate turn out to be low. This result also seems to be consistent with the popular recommendations from the analysts that the bond-stock ratio should increase as the time to horizon shortens. Type 1 agents will tend to hold less in bonds over time due to the decreasing risk premium offered by bonds. It is interesting to note that in the kind of equilibrium we employed, since bonds are in zero net supply, the representative agent will hold no bonds at all. Thus, if all investors have the same horizon, there must exist different investors who behave in opposite ways in the bond market.

Although our assumption of one-factor short rate process is quite restrictive, and may be unrealistic, there should be no conceptual difficulties to extend our framework to other term structure of interest rate models. This extension will inevitably require some cumbersome work and we conjecture that the qualitative properties of our results will be preserved. Moreover, the introduction of short sale constraint also seems unlikely to change the qualitative properties.



$\gamma$	$\eta$								
	2	3	4	5	6	7	8	9	10
2	0.00	0.85	1.27	1.52	1.69	1.82	1.91	1.98	2.03
3	-1.76	0.00	0.88	1.41	1.76	2.02	2.20	2.35	2.47
4	-3.86	-1.29	0.00	0.77	1.29	1.65	1.93	2.14	2.31
5	-6.04	-2.68	-1.01	0.00	0.67	1.15	1.51	1.79	2.01
6	-8.25	-4.12	-2.06	-0.82	0.00	0.59	1.03	1.37	1.65
7	-10.47	-5.59	-3.14	-1.68	-0.70	0.00	0.52	0.93	1.26
8	-12.71	-7.06	-4.24	-2.54	-1.41	-0.61	0.00	0.47	0.85
9	-14.95	-8.54	-5.34	-3.42	-2.14	-1.22	-0.53	0.00	0.43
10	-17.20	-10.03	-6.45	-4.30	-2.87	-1.84	-1.07	-0.48	0.00

Table 4.3: Optimal holding in the  $T$ -bond for a Type 2 agent.  $\gamma$  denotes the representative agent’s RRA coefficient and  $\eta$  denotes the non-representative agent’s RRA coefficient.

$\gamma$	$\eta$								
	2	3	4	5	6	7	8	9	10
2	0.00	0.33	0.50	0.60	0.67	0.71	0.75	0.78	0.80
3	-0.50	0.00	0.25	0.40	0.50	0.57	0.63	0.67	0.70
4	-1.00	-0.33	0.00	0.20	0.33	0.43	0.50	0.56	0.60
5	-1.50	-0.67	-0.25	0.00	0.17	0.29	0.38	0.44	0.50
6	-2.00	-1.00	-0.50	-0.20	0.00	0.14	0.25	0.33	0.40
7	-2.50	-1.33	-0.75	-0.40	-0.17	0.00	0.13	0.22	0.30
8	-3.00	-1.67	-1.00	-0.60	-0.33	-0.14	0.00	0.11	0.20
9	-3.50	-2.00	-1.25	-0.80	-0.50	-0.29	-0.13	0.00	0.10
10	-4.00	-2.33	-1.50	-1.00	-0.67	-0.43	-0.25	-0.11	0.00

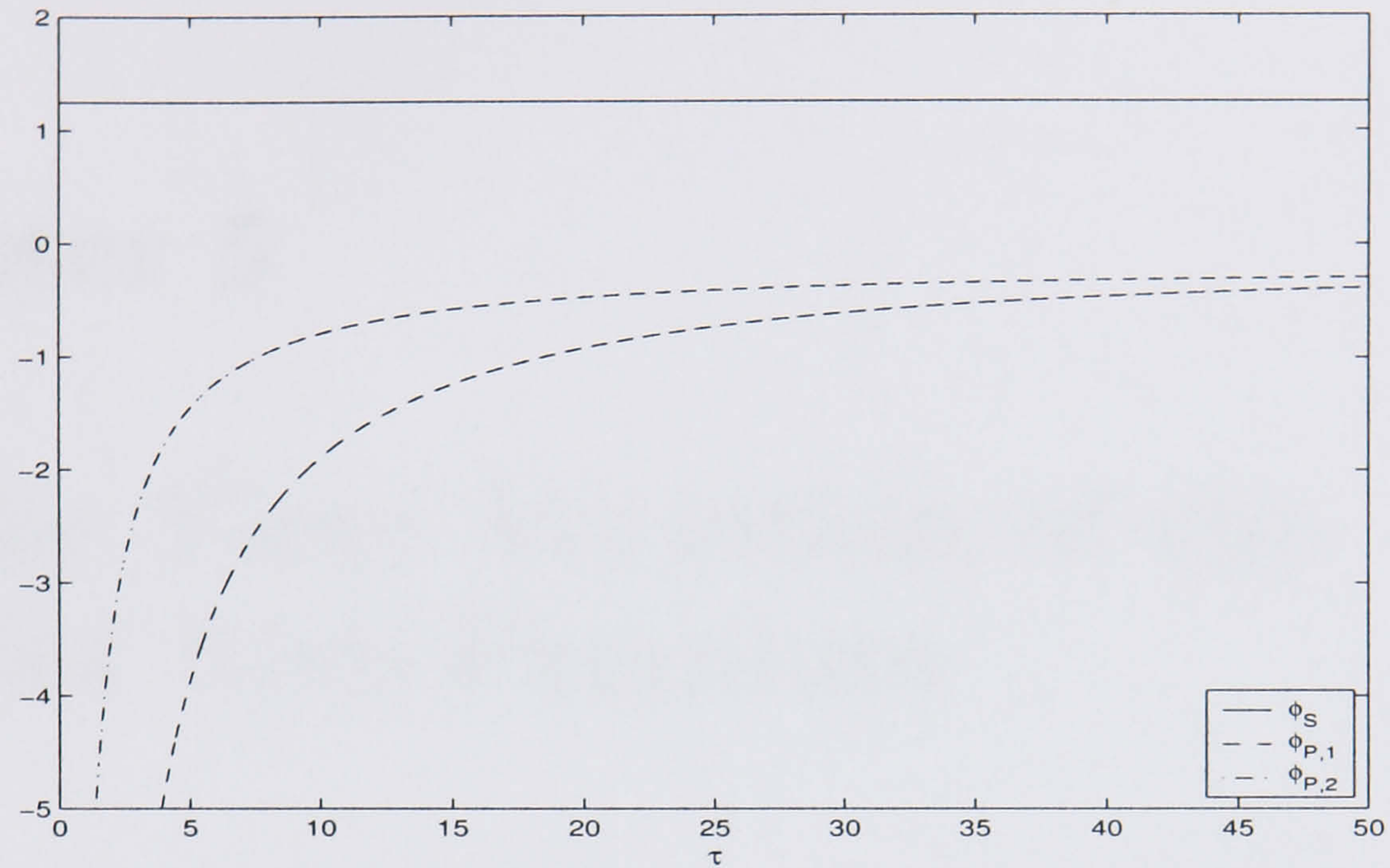
Table 4.4: Optimal holding in the  $T$ -bond for the limit case as  $\tau \rightarrow \infty$ .  $\gamma$  denotes the representative agent’s RRA coefficient and  $\eta$  denotes the non-representative agent’s RRA coefficient.



In view of the fundamental role the interest rate plays in the intertemporal analysis, it will be most interesting to see how the agent's optimal portfolio choice will behave with the introduction of intermediate consumption.



Panel I



Panel II

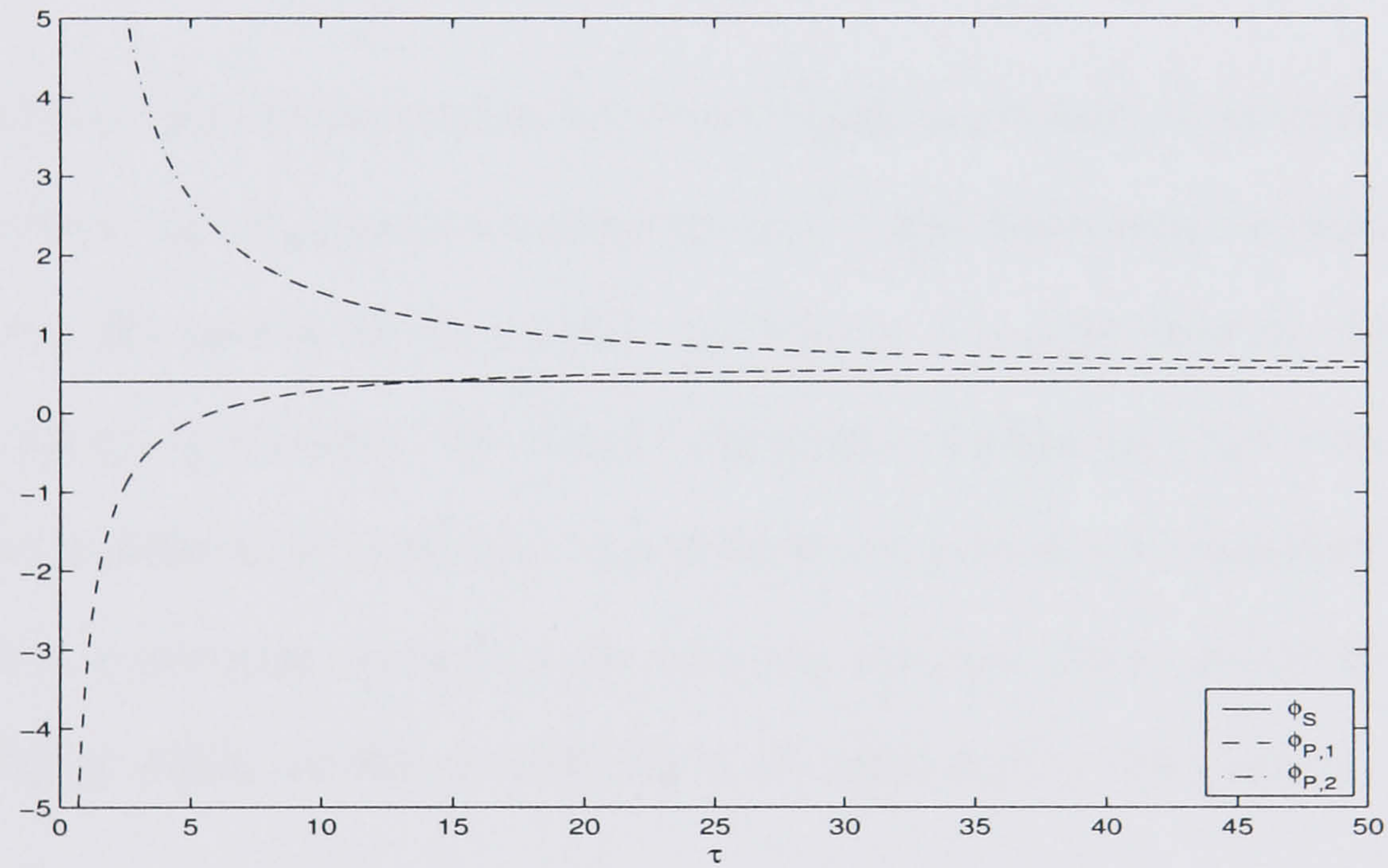


Figure 4.3: Optimal portfolio choice across time. These figures show the optimal holdings in the stock index and the  $T$ -bond across different horizons. Panel I shows the result for the case where  $\gamma = 3$  and  $\eta = 2$ . Panel II shows the result for the case where  $\gamma = 2$  and  $\eta = 5$ . The solid lines represent the stock index holdings, the dashed lines represent the  $T$ -bond holdings of a Type 1 agent, and the dash-dot lines represent the  $T$ -bond holdings of a Type 2 agent. The parameter values are  $\lambda = 0.06$ ,  $\sigma = 0.2$ ,  $b = 0.02$ , and  $\rho = -0.25$ .



# Chapter 5

## On the Time Variation of the Market Risk Premium

Motivated by the empirical observation that there exists some degree of predictability in asset returns, this chapter investigates the theoretical constraints on the time variation in the risk premia of the market portfolio in a continuous-time, finite horizon pure exchange economy. By characterising the equilibrium conditions as nonlinear partial differential equations, closed-form solutions can be obtained. It is shown that in a stationary economy, the presence of intermediate consumption can have a drastic effect on the possible kinds of time-varying behaviour of the risk premia.<sup>1</sup>

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<sup>1</sup>This chapter is a modified version of an earlier working paper presented in the AFFI 2002 Annual Conference, Strasburg, France, June 2002.



## 5.1 Introduction

The classical *Gordon growth model* (Gordon [49]) is one of the simplest theoretical models in finance. Nevertheless, it provides some basic economic intuitions for the equilibrium relations between prices, returns and dividends. First, the stock prices are high when dividends are expected to grow rapidly or when dividends are discounted at a low rate. Furthermore, the dividend-price ratio has a strong relationship with the prospective stock return: for a given growth rate, the higher the ratio, the higher the expected total return.

A substantial body of empirical literature on market returns often suggest that they exhibit certain degree of predictability for long horizons (see, for example, Poterba and Summers [92], Cecchetti, Lam and Mark [22], Kandel and Stambaugh [71], and Fama and French [46, 47]). However, the relaxation of the constancy of the discount rate and/or the dividend-price ratio very often introduces nonlinearity in the model and, therefore, becomes quite difficult to deal with.

One of the most contentious phenomena in the equity markets is the mean reversion in equity prices or returns. Since expected returns are not observable, we cannot be sure about whether the mean reversion is due to market inefficiency, or it should be attributed to the equilibrium asset pricing model. Nonetheless, the work by Cecchetti, Lam and Mark [22] indicates that mean reversion could be consistent with equilibrium. Other researchers have also tried to explain mean reversion by applying habit formation models, or more general utility functions such as the Epstein-Zin utility.

In the continuous-time literature, it is also not uncommon to find that a mean-reverting type of stochastic process is explicitly assumed and incorporated in the asset price dynamics in order to solve various problems. For example, the price of risk can be prespecified as following a mean-reverting stochastic process (see Black [6]). However, the way that the behaviour is modelled *exogenously* does not seem to be satisfactory. As in Bick [4, 5], not any given diffusion process can be used to represent the price process of the market portfolio. In other words, they might not be *viable*. Cox, Ingersoll and Ross [27, 28] also address the potential problem of an exogenously specified stochastic price process in that it may be internally inconsistent with an equilibrium.

For a price process to be viable, certain conditions must be satisfied and the utility function of the representative agent must also be identified in order to correspond to the given process in equilibrium. Along this line of research, some progress has been made (see for example, Bick [5], He and Leland [57], Hodges and Carverhill [61], and Hodges and Selby [63]). The equilibrium behaviour of the market parameters is better understood under this approach. Yet, these models fail to suggest that a diminishing relative risk aversion, desirable on empirical ground, could exist in equilibrium. Since these models assume the representative agent maximises her expected utility over terminal wealth only, it is natural to extend to a model which also considers intermediate consumption.<sup>2</sup>

Specifically, the purpose of this chapter is to investigate the theoretical con-

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<sup>2</sup>Although Bick [4] generalises his analysis on the consistency of the Black-Scholes model to a continuous-consumption framework, he assumes the proportional dividend model of Merton [86] and, therefore, can not generate a time-varying price of risk.



straints on the time variation in the risk premia of the market portfolio in an exchange economy of Lucas [70]. We follow the above line of research and extend the analysis one step further to the more complicated case where the representative agent demands intermediate consumption as well as the consumption at the horizon date. The preferences of the representative agent are represented by a state-independent von Neumann-Morgerstern utility function over consumption (either the date- $T$  only consumption or the continuous consumption)<sup>3</sup>. With continuous consumption, the role of dividends comes to play quite naturally within the single representative agent framework. We demonstrate how one can construct a model of such an economy in which security prices display mean reversion (the representative agent has diminishing relative risk aversion).<sup>4</sup>

The remainder of this chapter is organised as follows. Section 5.2 describes the conventional setting of a single representative agent economy and outlines respectively the formulation for both the case without intermediate consumption and with it. Section 5.3 then provides the characterisation of equilibrium asset price processes for both cases. The main result, given in Section 5.4, is a set of closed form solutions for a subclass time-homogeneous diffusion processes in a Black-Scholes economy. Numerical examples are given in Section 5.5 to demonstrate the behaviour of both price of risk and dividend yield. Section 5.6 concludes.

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<sup>3</sup>As we mentioned earlier, such a construction of utility function is a rather restrictive representation since it is well known from the work of Harrison and Kreps [56] that any arbitrage-free price system can be sustained in a competitive equilibrium with a representative agent whose preferences are defined by the price functional.

<sup>4</sup>As mentioned by a remark made by Campbell, Lo and MacKinlay [20], there is an increasing interest in the idea that risk aversion may vary over time with the state of the economy. They also address the prospect that the time-varying risk aversion might be able to explain the large body of evidence that excess returns on stocks and other risky assets are predictable.

## 5.2 The Formulation

We consider a continuous-time, finite horizon, pure exchange economy of Lucas [70]. The financial markets are assumed to be complete and we assume that the economy can be described by a single representative agent. The agent trades and acts as an expected utility maximiser and in equilibrium will optimally hold the market portfolio (representing the aggregate wealth of the whole economy) through time. There are two long-lived financial securities available for trading: a risky asset (the *stock*), and a locally riskless asset (the *bond*). At time  $t$ , the *trading price* of the stock is denoted by  $S_t$  and the holder of the stock is entitled to its dividends, if the stock is dividend-paying. The bond price is denoted by  $R_t$ , and increases at the instantaneous riskless rate of interest  $r_t$ . In equilibrium, there is one share of the stock outstanding and the bond is in zero net supply.

In the rest of this section, we shall describe two distinct economic settings. First, we consider an economy where the representative agent is concerned with her terminal wealth only. Second, we consider an economy where the representative agent concerns not only the terminal wealth but also intermediate consumption. We will refer to the former case as the *one-consumption* economy, and the latter one as the *continuous-consumption* economy. Finally, we discuss two important properties that arise from the first-order conditions and explain the significance of the path-independence result which leads us to the fundamental PDEs as the necessary conditions for equilibrium.



### 5.2.1 Without Intermediate Consumption

Consider a simple economy in which the only consumption good is produced exogenously and can only be consumed by individuals at time  $T$  when all the economic activities will end. There are two infinitely divisible securities that are continuously and frictionlessly traded in the market: a risky asset (the stock) that entitles the holder to the ownership of the consumption good at time  $T$ , and a riskless asset (the bond) that pays one unit of consumption at time  $T$ . The stock (the market portfolio) pays no dividends and its price will be endogenously determined. Suppose that  $S$  follows a diffusion process given by

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dz_t, \quad (5.1)$$

where  $z$  is the standard Brownian motion under the objective probability measure  $\mathbb{P}$  and the drift  $\mu$  and the diffusion  $\sigma$  are deterministic functions of  $S$  and  $t$ . Denote by  $\alpha$  the price of risk that represents the instantaneous reward per unit of risk, i.e.

$$\alpha(S_t, t) = \frac{\mu(S_t, t) - r(S_t, t)}{\sigma(S_t, t)}, \quad (5.2)$$

where  $r$  is the instantaneous riskless interest rate and is assumed to be a deterministic function of  $S_t$  and  $t$  as well. Hence, we can rewrite (5.1) as

$$\frac{dS_t}{S_t} = [r(S_t, t) + \sigma(S_t, t) \cdot \alpha(S_t, t)] dt + \sigma(S_t, t)dz_t. \quad (5.3)$$

Moreover, the bond price  $B$  accumulates at the riskless rate  $r$ :

$$\frac{dB_t}{B_t} = r_t dt. \quad (5.4)$$

Assume that the agent is endowed with a positive initial amount but receives no intermediate income. In addition, consumption only occurs at the agent's in-

vestment horizon date  $T$ , same as that of the economy. She then aims to optimally allocate her wealth in the stock and the bond in order to maximise her expected utility over the time- $T$  wealth,  $W_T$ :

$$\max \quad \mathbb{E}[U(W_T)], \quad (5.5)$$

where  $U$  is a strictly increasing, state-independent, and continuously differentiable von Neumann-Morgenstern utility function.

Denote by  $\Phi$  the amount of money invested in the stock. Then the wealth function  $W$  follows the process:

$$dW_t = [W_t r_t + \Phi_t(\mu_t - r_t)] dt + \Phi_t \sigma_t dz, \quad (5.6)$$

with  $W_0 = w > 0$  (positive initial wealth) and  $W_t \geq 0$  (nonnegative wealth constraint), for  $0 < t \leq T$ .

Following Harrison and Kreps [56], it is well known that in any arbitrage-free pricing system there exists a risk-neutral probability measure under which the drift of the stock returns is the riskless rate  $r$ . Hence we can let  $\mathbb{Q}$  denote the risk-neutral probability measure and  $\mathcal{M} = d\mathbb{Q}/d\mathbb{P}$  denote the change of measure (i.e. the *Radon-Nykodym derivative*) from  $\mathbb{P}$  to  $\mathbb{Q}$ . The state-price density (SPD) can then be defined as

$$\xi_t = e^{-\int_0^t r_s ds} \cdot \mathcal{M}_t.$$

Since in equilibrium the representative agent will be not to trade at all and should hold the market portfolio, the agent's marginal utility can be related to the SPD through the first order condition

$$\frac{\partial U(S_T)}{\partial S_T} = \lambda \cdot \xi_T. \quad (5.7)$$



where  $\lambda$  is the Lagrange multiplier.

### 5.2.2 With Intermediate Consumption

If, alternatively, we allow the presence of intermediate consumption and assume the stock pays dividends, the *trading price* of the stock  $S$  can be formulated as:

$$dS_t = [\mu(S_t, t)S_t - D(S_t, t)] dt + \sigma(S_t, t)S_t dz_t, \quad (5.8)$$

where  $z$  is the standard Brownian motion under the  $\mathbb{P}$ -measure and  $D$  is the cash dividend paid by the stock and is assumed to be a deterministic function of  $S$  and  $t$ . Similarly, (5.8) can be rewritten as

$$dS_t = [(r(S_t, t) + \sigma(S_t, t) \cdot \alpha(S_t, t))S_t - D(S_t, t)] dt + \sigma(S_t, t)S_t dz_t, \quad (5.9)$$

where  $r$ ,  $\sigma$ , and  $\alpha$  are as defined before.

The agent's maximisation problem is now formulated as follows:

$$\max \quad \mathbb{E}_0 \left[ \int_0^T e^{-\rho t} U_1(C_t) dt + U_2(W_T) \right], \quad (5.10)$$

where  $\rho$  is the rate of time preference and  $U_1$  and  $U_2$  are strictly increasing, time additive, and state-independent von Neumann-Morgenstern utility functions and are continuously differentiable, where applicable.

Again let  $\Phi_t$  denote the amount of money invested in the stock at the beginning of time- $t$  period and  $C_t$  the amount being consumed during time  $t$ -period. Thus the investor's indirect utility (or value function) can be defined as:

$$J(W_t, S_t, t) = \max_{\Phi, C} \quad \mathbb{E}_t \left[ \int_t^T e^{-\rho(s-t)} U_1(C_s) ds + U_2(W_T) \right], \quad (5.11)$$

where  $J(W_T, S_T, T) = U_2(W_T)$  is the boundary condition. The wealth process follows:

$$dW_t = \left[ W_t r_t + \Phi_t(\mu_t - r_t) - \left( \frac{\Phi_t}{S_t} \right) C_t \right] dt + \Phi_t \sigma_t dz, \quad (5.12)$$

where  $W_0 = w > 0$  (positive initial wealth) and  $W_t \geq 0$  (nonnegative wealth constraint), for  $0 < t \leq T$ .

Recall that in equilibrium the representative agent's optimal strategy is to hold the market portfolio and consume all the dividends received from the stock investment. Hence, the first order condition of optimality for the Hamilton-Jacobi-Bellman (HJB) equation yields:

$$e^{-\rho t} \frac{\partial U_1(C_t)}{\partial C_t} = \frac{\partial J(W_t, S_t, t)}{\partial W_t} = \lambda \cdot \xi_t, \quad (5.13)$$

where  $\lambda$  is the Lagrange multiplier. In equilibrium,  $C_t = D_t$  and  $W_t = S_t$  for  $t < T$ .

### 5.2.3 Monotonicity Property and the Path Independence Result

As we have emphasised before, an efficient trading strategy must be path-independent and generate a payoff that is monotonically increasing in the market level, at least in our construction of economy. These properties, of course, must apply to the market portfolio itself too when we assume that the market is always efficient.

Let us briefly recall these results. By inspection of the first-order conditions (5.7) and (5.13), we first note that with the assumption of increasing utility functions, wealth should be monotonically and inversely related to the marginal utility



(or the SPD). This property applies to both cases as it can be seen that from (5.7),  $W_T = V(\lambda \cdot \xi_T)$ , where  $V$  is the inverse function of the marginal utility  $U'$ , and from (5.13),  $W_t = I(\lambda \cdot \xi_t)$ , where  $I$  is the inverse function of the marginal indirect utility  $J'$ . Thus a portfolio strategy which creates state-dependent wealth will be called an *efficient* strategy only if the monotonicity property is satisfied (see Dybvig [36, 37]).

In the context of equilibrium, since the stock price  $S_t$  represents the agent's wealth and the monotonicity property should apply, thus at each point of time  $t \in [0, T]$ , the process of the SPD  $\xi_t$  must be path-independent, regardless the stock price history (see Cox and Leland [29, 30]). It is this very result that paves the way for us to analyse the equilibrium asset price dynamics in the economy.<sup>5</sup>

## 5.3 Equilibrium Conditions of the Asset Prices Dynamics

In this section, we shall provide the characterisation of equilibrium price processes for both the one-consumption economy and the continuous-consumption economy. In each case, we shall first derive a general partial differential equation for the *intertemporal relative risk aversion*  $f$  with respect to  $S$ . For simplicity, we shall assume the constancy of  $r$  and  $\sigma$  and translate the PDE to an equivalent one in terms of the *price of risk*  $\alpha$  with respect to a new transformed variable  $x$ , where

$$x_t = \ln S_t - \left( r - \frac{\sigma^2}{2} \right) t.$$

---

<sup>5</sup>See for example, Hodges and Carverhill [61] and He and Leland [57]. These authors also characterise the equilibrium price processes by exploiting this property.

### 5.3.1 Without Intermediate Consumption

**Theorem 5.1 (Equilibrium conditions: without consumption)** *Assume in the economy, there exists one non-dividend-paying risky asset (the stock) and one riskless asset (the money account or the bond). The representative agent continuously allocates her wealth among these two assets according to her objective function (5.5) subject to the wealth process (5.6) and then consumes her terminal wealth at time  $T$ . The necessary condition for the asset price dynamics (5.1) to be an equilibrium process when  $r$ ,  $\mu$  and  $\sigma$  are deterministic functions of  $S$  and  $t$  is that the coefficients must satisfy the following PDE:*

$$\mathcal{L}f + f_t + r_S S(f - 1) + \sigma \sigma_S S(S f_S + f^2 - f) = 0, \quad (5.14)$$

where

$$\begin{aligned} f(S_t, t) &= \frac{\mu(S_t, t) - r(S_t, t)}{(\sigma(S_t, t))^2}, \\ \mathcal{L}f &= \frac{1}{2} \sigma^2 S^2 f_{SS} + \mu S f_S, \end{aligned}$$

and the boundary condition is

$$f(S_T, T) = -S_T \frac{U''(S_T)}{U'(S_T)}.$$

**Proof:** The main idea is to exploit the path-independence property on the state-price density. Recall that the process of  $\xi$  is defined as

$$\xi_t = \exp \left( - \int_0^t r_s ds - \int_0^t \left( \frac{\mu_s - r_s}{\sigma_s} \right) dz_s - \frac{1}{2} \int_0^t \left( \frac{\mu_s - r_s}{\sigma_s} \right)^2 ds \right).$$

Now, define a new variable  $Z(S_t, t) = \ln \xi(S_t, t)$ . We then apply Itô's Lemma to derive  $dZ$  and equate it with  $d(\ln \xi)$ . Collecting  $dt$  and  $dz$  terms respectively



yields the following equations:

$$\begin{aligned} Z_t + \mu S Z_S + \frac{1}{2} \sigma^2 S^2 Z_{SS} &= -r - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 = -r - \frac{1}{2} \sigma^2 f^2, \\ \sigma S Z_S &= - \left( \frac{\mu - r}{\sigma} \right) = -\sigma f. \end{aligned}$$

Note that for notational simplicity, we have suppressed the time index so that it will not be confused with the partial derivatives.

From the second equation above, we can derive  $Z_{St} = -\frac{f_t}{S}$ ,  $Z_S = -\frac{f}{S}$  and  $Z_{SS} = -\frac{f_S}{S} + \frac{f}{S^2}$ . Substitute  $Z_S$  and  $Z_{SS}$  into the first equation and use  $\mu = r + \sigma^2 f$  to obtain

$$Z_t = -r + rf + \frac{1}{2} \sigma^2 f^2 + \frac{1}{2} \sigma^2 S f_S - \frac{1}{2} \sigma^2 f.$$

Differentiate the above equation with respect to  $S$  and equate it with  $Z_{St}$  to obtain

$$Z_{tS} = r_S(f - 1) + (r + \sigma^2 f)f_S + \frac{1}{2} \sigma^2 S f_{SS} + \sigma \sigma_S(S f_S + f^2 - f) = Z_{St} = -\frac{f_t}{S}.$$

The result (5.14) then immediately follows.  $\square$

Theorem 5.1 states the general equilibrium conditions which the *intertemporal relative risk aversion*  $f$  must satisfy. We now turn to some special cases. The first case is a Black-Scholes economy where both the interest rate  $r$  and the volatility of the stock return  $\sigma$  are constant. Therefore, by letting  $\sigma_S = 0$  and  $r_S = 0$ , (5.14) can be simplified as

$$\mathcal{L}f + f_t = 0. \tag{5.15}$$

The strong assumption of constancy of  $r$  and  $\sigma$  enables us to obtain a nice result known as the *Burgers' equation*. The finance application of this equation

seems to first appear in Hodges and Carverhill [61] and in an independent work of He and Leland [61]. For completeness, we recite the result in the following theorem.

**Theorem 5.2 (Burgers' equation)** *Assume constant  $r$  and  $\sigma$  and define the transformed state variable  $x$  as  $x_t = \ln S_t - (r - \sigma^2/2)t$ . The the price of risk  $\alpha$  in the wealth-only economy must evolve over time according to the PDE:*

$$\alpha_t = \frac{1}{2}\sigma^2\alpha_{xx} + \sigma\alpha\alpha_x. \quad (5.16)$$

**Proof:** By definition, we have  $\mu - r = \sigma^2 f = \sigma\alpha$  and  $x = \ln S - \left(r - \frac{\sigma^2}{2}\right)t$ .

Thus, we can write

$$\alpha(S, t) \equiv \alpha(x, t) = \sigma \cdot f\left(e^{x + \left(r - \frac{\sigma^2}{2}\right)t}, t\right)$$

and its partial derivatives

$$\begin{aligned} \alpha_x &= \sigma S f_S, \\ \alpha_{xx} &= \sigma(f_S + S f_{SS})S \\ \alpha_t &= \sigma f_S \left(r - \frac{\sigma^2}{2}\right) S + \sigma f_t. \end{aligned}$$

Rearranging the above equations to obtain  $f_S$ ,  $f_{SS}$  and  $f_t$  and substituting them into (5.15) yields

$$\alpha_t + \sigma\alpha\alpha_x + \frac{1}{2}\sigma^2\alpha_{xx} = 0.$$

This immediately gives (5.16).  $\square$

Another interesting case is to assume that  $f$  is time-invariant in the sense that  $\mu$ ,  $r$ , and  $\sigma$  are functions of  $S$  only. We then obtain the following proposition:



**Proposition 5.1** *When  $\mu$ ,  $r$ , and  $\sigma$  are functions of  $S$  only, the equilibrium condition for the economy stated in Theorem 5.1 is that the following equation must be satisfied:*

$$\frac{\partial}{\partial S}[\sigma^2(Sf_S + f^2 - f) + 2r(f - 1)] = 0. \quad (5.17)$$

*Or equivalently, there exists a constant  $K$  such that*

$$\sigma^2(Sf_S + f^2 - f) + 2r(f - 1) = K.$$

**Proof:** By assumption,  $\mu = r + \sigma^2 f$  and  $f_t = 0$ . After some simple manipulation, (5.17) can then be easily derived from (5.14):

$$\begin{aligned} 0 &= \mathcal{L}f + f_t + r_S S(f - 1) + \sigma \sigma_S S(Sf_S + f^2 - f) \\ &= \frac{1}{2} \sigma^2 S^2 f_{SS} + (r + \sigma^2 f) S f_S + S \frac{\partial}{\partial S} [r(f - 1)] - r S f_S \\ &\quad + \frac{1}{2} S(Sf_S + f^2 - f) \frac{\partial \sigma^2}{\partial S} \\ &= \frac{1}{2} S \sigma^2 \frac{\partial}{\partial S} (Sf_S + f^2 - f) + S \frac{\partial}{\partial S} [r(f - 1)] + \frac{1}{2} S(Sf_S + f^2 - f) \frac{\partial \sigma^2}{\partial S} \\ &= \frac{1}{2} S \frac{\partial}{\partial S} [\sigma^2(Sf_S + f^2 - f) + 2r(f - 1)] . \square \end{aligned}$$

**Remark 5.1** *In the same setting as ours, except assuming constant interest rate, He and Leland ([57], pp. 603-604) provide a similar necessary condition for the time-invariant case, namely,  $\sigma^2(f^2 + Sf_S - f) = K$ . Therefore, it must be pointed out that their condition strictly only holds when  $r = 0$ .*

*Their result can be justified if we define  $S$  as the **relative** asset price (that is, the risky asset price normalised by the bond price). The drift term  $\mu$  in PDE (5.14) should then be interpreted as the **risk premium**, provided that the risk premium*

*is a deterministic function of the relative price and time. This is in fact the setup in Bick [5].*

Without further assumptions, the PDE (5.14) is difficult to solve in general. To date, several functional form solutions to the time-invariant case with constant  $\sigma$  and  $r = 0$  can be found as examples in Bick [5] and in He and Leland [57]. For a more general definition on time independence of the diffusion processes, Hodges and Selby [63] carried out a time-homogeneous analysis for the case with constant volatility and constant interest rate. They seek to find steady-state solutions to the Burgers' equation (5.16) by constraining the risk premium to vary depending on the level of the market in such a way that the functional form does not depend on time. Interestingly, they conclude that there are only two possible viable solutions and one non-viable one for the steady state: the price of risk can be constant or increasing in aggregate wealth, but the only steady state solution with decreasing price of risk admits arbitrage (and is not viable).

The finding of an increasing price of risk is somewhat disappointing as it would be nice if we can produce a mean-reverting behaviour in such a simple model. Nevertheless, it is conceivable that the introduction of intermediate consumption might be sufficient to modify this behaviour. As we shall illustrate in the next section, it is indeed the case: with large enough intermediate consumption, there exists a decreasing price of risk in the steady state which stems from decreasing relative risk aversion of the representative agent.



### 5.3.2 With Intermediate Consumption

Now we are to characterise the equilibrium conditions for the economy with intermediate consumption. Since it is more convenient to work with the dividend yield rather than the dividend amount, we denote by  $\delta$  the dividend yield to mean  $\delta(S_t, t) = D(S_t, t)/S_t$ . The next two theorems generalise on equations (5.14) and (5.16) to include dividends. The approach is similar to that used before.

**Theorem 5.3 (Equilibrium conditions: with consumption)** *Assume in the economy, there exists one dividend-paying stock and one riskless bond. The representative agent continuously allocates her wealth among the two assets according to her objective function (5.10) subject to the wealth process (5.12) and consumes the dividends paid by the stock investment. The necessary condition for the asset price dynamics (5.8) to be an equilibrium process when  $r$ ,  $\mu$ ,  $\delta$  and  $\sigma$  are deterministic functions of  $S$  and  $t$  is that the coefficients must satisfy the following PDE:*

$$\mathcal{L}f + f_t - \delta_S S f + r_S S(f - 1) + \sigma \sigma_S S(S f_S + f^2 - f) = 0, \quad (5.18)$$

where

$$\begin{aligned} f(S_t, t) &= \frac{\mu(S_t, t) - r(S_t, t)}{(\sigma(S_t, t))^2} = -\frac{U_1''(D(S_t, t)S_t D_S(S_t, t))}{U_1'(D(S_t, t))}, \\ \mathcal{L}f &= \frac{1}{2}\sigma^2(S_t, t)S^2 f_{SS}(S_t, t) + (\mu(S_t, t) - \delta(S_t, t))S f_S(S_t, t), \end{aligned}$$

and the boundary condition is given by

$$f(S_T, T) = -S \frac{U_2''(S_T)}{U_2'(S_T)}.$$

**Proof:** This is simply a rederivation of Theorem 5.1 with the presence of intermediate consumption (dividends). Again define a new variable  $Z(S_t, t) = \ln \xi(S_t, t)$ .

We apply *Ito's Lemma* to derive  $dZ(S_t, t)$  and equate it with  $d(\ln \xi)$ . Collecting  $dt$  and  $dz$  terms respectively yields

$$\begin{aligned} Z_t + (\mu S - D)Z_S + \frac{1}{2}\sigma^2 S^2 Z_{SS} &= -r - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 = -r - \frac{1}{2}\sigma^2 f^2, \\ \sigma S Z_S &= -\left(\frac{\mu - r}{\sigma}\right) = -\sigma f. \end{aligned}$$

Following the same technique in the proof of Theorem 5.1 and applying  $\delta = \frac{D}{S}$ . (5.18) can then be easily obtained.  $\square$

Theorem 5.3 provides the equilibrium conditions for the asset price processes to satisfy in an economy with intermediate consumption. It is rather difficult to obtain an explicit solution for the PDE (5.18). However, if we assume the constancy of  $r$  and  $\sigma$ , the problem can be greatly simplified. The following theorem provides an equation which is analogous to the Burgers' equation in Theorem 5.2 but with some additional terms.

**Theorem 5.4** *Assume  $r$  and  $\sigma$  are constant and define the transformed state variable  $x$  as  $x_t = \ln S_t - (r - \sigma^2 t)/2$ . Then the price of risk  $\alpha$  in the consumption economy must evolve over time according to the PDE:*

$$\alpha_\tau = \frac{1}{2}\sigma^2 \alpha_{xx} + \sigma \alpha \alpha_x - \delta \alpha_x - \delta_x \alpha. \quad (5.19)$$

**Proof:** Apply  $r_S = 0$  and  $\sigma_S = 0$  to (5.18) and rearrange to obtain

$$f_t + \frac{1}{2}\sigma^2 S^2 f_{SS} + \mu S f_S - \delta S f_S - \delta_S S f = 0. \quad (5.20)$$



By definition,  $\sigma f = \alpha$  and  $x = \ln S - (r - \sigma^2 t)/2$ . Thus, (5.19) immediately follows by substituting  $\sigma S f_S = \alpha_x$ ,  $\delta_S S = \delta_x$  and

$$f_t + \frac{1}{2}\sigma^2 S^2 f_{SS} + \mu S f_S = \frac{1}{\sigma} \left( \alpha_t + \sigma \alpha \alpha_x + \frac{1}{2}\sigma^2 \alpha_{xx} \right)$$

into (5.20).  $\square$

## 5.4 The Time-Homogeneous Case in a Black-Scholes Economy

In this section, we study an important special case of the time homogeneous economy. We seek to find the steady-state solutions to the PDE (5.19) in order to see how the price of risk and the dividend (consumption) vary depending on the market level in such a way that the functional forms are independent of time.

We start by specifying two functions, the price of risk  $\alpha$  and the dividend yield  $\delta$ , which together satisfy the PDE (5.19). By homogeneity in time, we mean that  $\alpha$  and  $\delta$  can be specified as functions of the change of variable  $u$ . Specifically, the two functions are represented as follows:

$$\alpha(x, \tau) \equiv y(u), \tag{5.21}$$

$$\delta(x, \tau) \equiv g(u), \quad \text{where } u = x + \theta\tau, \tag{5.22}$$

for some functions  $y$  and  $g$  and some constant  $\theta$ .

Thus, the partial derivatives in (5.19) can be expressed by  $\alpha_\tau = \theta y'$ ,  $\alpha_x = y'$ ,  $\alpha_{xx} = y''$  and  $\delta_x = g'$ . It follows that the PDE (5.19) can be reduced to an ODE:

$$-\theta y' + \sigma y y' + \frac{1}{2}\sigma^2 y'' - g y' - g' y = 0. \tag{5.23}$$

The above equation is difficult to solve, in general, as it involves the joint behaviour of  $y$  and  $g$ . We can, of course, view (5.23) as an ODE in  $y$  given a  $g$  function, but then the question will be how we should choose  $g$ . Since  $g$  could be a function of a rather arbitrary form, a more plausible question might be to ask whether there are cases which permit closed-form solutions.

For simplicity, we assume that the dividend yield function  $g$  is an affine function with respect to the risk premium of the market portfolio. Recall that  $y = \mu/\sigma$ , namely, the risk premium  $\mu$  is equal to  $\sigma y$ . In other words, our assumption amounts to the following equality

$$g(u) = p_0 + p_1 \sigma y(u), \quad (5.24)$$

for some constants  $p_0$  and  $p_1$ .

By imposing this simplifying assumption, (5.23) can be conveniently reduced to the following equation:

$$-(\theta + p_0)y' + (1 - 2p_1)\sigma y y' + \frac{1}{2}\sigma^2 y'' = 0. \quad (5.25)$$

At this point, it is worth noting that the linear form assumed in (5.24) is in the spirit of a first order Taylor Series approximation. Given that both variables  $g$  and  $y$  are likely to have limited ranges, and a monotonic relationship to each other (through their relation to the market level), we expect this to be a satisfactory representation. Nevertheless, in principle, we have selected a small subclass of the infinitely many possible dividend yield functions in order for us to obtain an analytic solution.

In the rest of this section, we shall provide the possible mathematical solutions



to (5.25) and discuss how they relate to the economics. In many cases, the solutions will be ruled out due to their infeasibility.

To solve  $y$ , we first integrate (5.25) once to obtain

$$\frac{1}{2}\sigma^2 y' = (\theta + p_0)y - \frac{1}{2}(1 - 2p_1)\sigma y^2 + \text{constant}. \quad (5.26)$$

The derivations of the solutions of (5.26) are rather lengthy and complex at times.

For ease of exposition, we shall discuss them in terms of cases as follows.

**Case 5.1** *When  $p_0 = -\theta$  and  $p_1 = 1/2$ , the price of risk  $y$  has a linear form solution:*

$$y(u) = k_1 u + k_2. \quad (5.27)$$

*where  $k_1$  and  $k_2$  are constants. Provided  $k_2 > 0$ ,  $y$  can be constant if  $k_1 = 0$ , or decreasing (increasing) if  $k_1 < 0$  ( $k_1 > 0$ ).*

**Proof:** In this case, (5.26) reduces to

$$\frac{1}{2}\sigma^2 y' = \text{constant}.$$

Rearrange to obtain

$$\frac{1}{2}\sigma^2 \int \frac{dy}{\text{constant}} = \int du.$$

It then integrates to (5.27).  $\square$

Recall that  $u = x + \theta\tau$ . Thus, a limitation of this linear solution is that except for the constant case (i.e.  $k_1 = 0$ ),  $y$  is unbounded above and below. Thus,

unless we impose additional constraint for  $x$  to stay within a certain range, the risk premium will go negative when the market level is large or small (depending on the sign of  $k_1$ ).

**Case 5.2** When  $p_0 \neq -\theta$  and  $p_1 = 1/2$ , the price of risk  $y$  has an exponential form solution:

$$y(u) = e^{m(u)} + L, \quad (5.28)$$

where

$$m(u) = k + \frac{2(\theta + p_0)}{\sigma^2} u,$$

and  $k$  and  $L$  are constants and  $L \geq 0$ . The price of risk  $y$  is decreasing (increasing) in  $x$  if  $p_0 < -\theta$  ( $p_0 > -\theta$ ).

**Proof:** In this case, (5.26) reduces to

$$\frac{1}{2}\sigma^2 y' = (\theta + p_0)y + \text{constant}.$$

Rearrange to obtain

$$\frac{\sigma^2}{2(\theta + p_0)} \int \frac{dy}{y + \text{constant}} = \int du.$$

Integration yields,

$$\frac{\sigma^2}{2(\theta + p_0)} \ln(y + c_1) = u + c_2, \quad \text{for } y \in (-c_1, +\infty) \quad (5.29)$$

where  $c_1$  and  $c_2$  are constants. The solution for  $y$  is of an exponential form,

$$y(u) = e^{m(u)} + L, \quad (5.30)$$



where

$$m(u) = k + \frac{2(\theta + p_0)}{\sigma^2}u,$$

and  $k$  and  $L$  are constants. Our assumption of a positive price of risk requires  $c_1 \leq 0$  in (5.29) and  $L \geq 0$  in (5.30). Finally, by differentiating  $y$  with respect to  $x$ , it follows that  $y$  is decreasing (increasing) in  $x$  if  $p_0 < -\theta$  ( $p_0 > -\theta$ ).  $\square$

The exponential solution is better than the linear one in that it ensures the risk premium to stay positive. However, the fact that it is unbounded above is still unfavourable.

**Case 5.3** When  $p_0 \neq -\theta$  and  $p_1 \neq 1/2$ , provided  $c_2$  is real, the price of risk  $y$  has a general solution

$$y(u) = \begin{cases} \frac{(c_2+c_1)e^{s(u)}+c_1-c_2}{e^{s(u)}+1} & \text{for } y \in (c_1 - c_2, c_1 + c_2) \\ \frac{(c_2+c_1)e^{s(u)}+c_2-c_1}{e^{s(u)}-1} & \text{for } y \notin (c_1 - c_2, c_1 + c_2) \end{cases} \quad (5.31)$$

where

$$c_1 = \frac{\theta + p_0}{\sigma(1 - 2p_1)}, \quad c_2 = \pm \sqrt{d + c_1^2}, \quad (5.32)$$

$s(u) = k + \frac{2(1-2p_1)c_2}{\sigma}u$ , and  $k$  is a constant. For  $y$  to be positive,  $c_1$  must be positive and greater than  $c_2$  if we take the positive value of  $c_2$ . When  $c_2 = 0$ ,  $y$  is constant and equal to  $c_1$ . Otherwise,  $y$  has a travelling wave solution from the first equation of (5.31) which is viable, and has a hyperbolic solution of the form,

$$\alpha(x, \tau) = c_1 + \frac{\sigma}{(1 - 2p_1)(x + \theta\tau)} \quad (5.33)$$

from the second equation of (5.31) which admits arbitrage.

**Proof:** In this case, we rearrange (5.26) to obtain

$$\frac{\sigma}{1-2p_1} \int \frac{dy}{d + \frac{2(\theta+p_0)}{\sigma(1-2p_1)}y - y^2} = \int du, \quad (5.34)$$

where  $d$  is a constant. Integrating (5.34) yields

$$\frac{\sigma}{2(1-2p_1)c_2} \ln \left[ \left| \frac{y + c_2 - c_1}{y - c_2 - c_1} \right| \right] = u + \text{constant},$$

where

$$c_1 = \frac{\theta + p_0}{\sigma(1-2p_1)}, \quad \text{and } c_2 = \pm \sqrt{d + c_1^2}.$$

Thus, provided  $c_2$  is real and not complex, we obtain a general solution of the form

$$y(u) = \begin{cases} \frac{(c_2+c_1)e^{s(u)}+c_1-c_2}{e^{s(u)}+1} & \text{for } y \in (c_1 - c_2, c_1 + c_2) \\ \frac{(c_2+c_1)e^{s(u)}+c_2-c_1}{e^{s(u)}-1} & \text{for } y \notin (c_1 - c_2, c_1 + c_2) \end{cases} \quad (5.35)$$

where  $s(u) = k + \frac{2(1-2p_1)c_2}{\sigma}u$  and  $k$  is a constant. While the first equation in (5.35) may result in hyperbolic tangent functions consistent with equilibrium, the second equation, in general, entails trigonometric functions which could not possibly be supported by any reasonable utility function of an economic agent. More specifically, the price of risk  $y$  has an unacceptable singularity at  $s(u) = 0$  for the second equation, except when  $c_2 = 0$  but  $k \neq 0$  which gives the trivial solution of a constant price of risk

$$y = c_1 = \frac{\theta + p_0}{\sigma(1-2p_1)}. \quad (5.36)$$

In other words, when  $c_2 \neq 0$ , the second equation of (5.35) prevents the state variable  $x$  from reaching the point

$$x = -\frac{k\sigma}{2(1-2p_1)c_2} - \theta\tau.$$



Another special case is to let  $k = 0$  and take the limit as  $c_2$  tends to zero. The solution will then be

$$y = c_1 + \frac{\sigma}{(1 - 2p_1)u},$$

which gives

$$\alpha(x, \tau) = c_1 + \frac{\sigma}{(1 - 2p_1)(x + \theta\tau)}, \quad (5.37)$$

It is clear that  $\alpha$  is decreasing (increasing) in  $x$  when  $p_1 < 1/2$  ( $p_1 > 1/2$ ). Unfortunately, (5.37) has a singularity at  $p_1 = 1/2$  and  $x = -\theta\tau$ . In other words, this model permits arbitrage in the economy and, therefore, must be ruled out.

The more interesting case is the stable travelling wave solution obtained from the first equation. By inspection of (5.36), it can be seen that if we assume a positive risk premium, then  $c_1 > 0$  would be required. That is, we have ruled out the possibility of  $p_0 = -\theta$  so that

$$\text{sign}(\theta + p_0) = \text{sign}(1 - 2p_1).$$

Provided that  $c_2$  is positive,  $c_1$  must be greater than  $c_2$  and we obtain two alternative scenarios:<sup>6</sup>

1. ( $p_0 > -\theta$  and  $p_1 < 1/2$ ) The solution  $y$  is increasing and is bounded below and above

$$\begin{cases} y \rightarrow c_1 - c_2, & \text{when } x \rightarrow -\infty \\ y \rightarrow c_1 + c_2, & \text{when } x \rightarrow +\infty \end{cases} \quad (5.38)$$

It is worth noting that the travelling wave solution (increasing in  $x$ ) Hodges and Selby [63] obtained is a special case in this scenario with  $p_0 = p_1 = 0$  (i.e., without the presence of intermediate consumption).

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<sup>6</sup>Note that taking a negative  $c_2$  will not change the properties of the solutions.

2. ( $p_0 < -\theta$  and  $p_1 > 1/2$ ) The solution  $y$  is decreasing and is bounded below and above

$$\begin{cases} y \rightarrow c_1 + c_2, & \text{when } x \rightarrow -\infty \\ y \rightarrow c_1 - c_2, & \text{when } x \rightarrow +\infty \end{cases} \quad (5.39)$$

□

Note that (5.39) demonstrates the kind of behaviour we are most interested in and more importantly, it can not be obtained without introducing intermediate consumption in the economy.<sup>7</sup>

## 5.5 Numerical Examples

In this section, we illustrate the analytical findings presented in Section 5.4 using an empirical parameterisation of the process for stock indices and the process for dividend yields. We first summarise the various possible patterns for the price of risk function, and then present and analyse the numerical results for the chosen model.

In Figure 5.1, typical plots are produced for the cases analysed in Section 5.4. Panel I shows the linear form solutions, Panel II shows the exponential form solutions, Panel III shows the travelling wave solutions, and finally Panel IV shows

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<sup>7</sup>It is also interesting to derive a  $y$  function which depends on  $S$  only. This can be achieved by letting  $\theta = -\left(r - \frac{\sigma^2}{2}\right)$  so that the resulting  $y$  can be expressed as

$$y = \frac{(c_2 + c_1)B \cdot S^A + c_1 - c_2}{B \cdot S^A + 1}, \text{ for } y \in (c_1 - c_2, c_1 + c_2).$$

where  $A = \frac{2(1-2p_1)c_2}{\sigma}$  and  $B$  is a positive real. Again, provided  $p_0 < -\theta$  and  $p_1 > 1/2$ , we can obtain a decreasing  $y$ .



the hyperbolic form solutions with singularity. The solid lines represent the patterns which can be obtained only when the intermediate consumption is present in the economy, whereas the dashed lines represent the patterns which can also be achieved when only the terminal wealth is being considered. Note the added flexibility which comes from including intermediate consumption. Additionally, the Panel IV case should be excluded since it admits arbitrage.

As has been described before, in our underlying economy, consumption equals dividend in equilibrium. Therefore, it seems desirable that the dividend yields should be (1) decreasing in the state of economy, (2) increasing in the price of risk, and (3) decreasing in the asset price (supposedly sticky). In addition, an increase in dividends should imply an increase in prices.

We can show that our model can successfully generate the above properties. For instance, the decreasing travelling wave solution of the price of risk  $h$ , as in (5.31) and (5.39), implies that the dividend yield  $g$  has a positive slope  $p_1$  and, therefore, is increasing in  $h$ .

Recall that  $p_1$  must satisfy the constraint ( $p_1 > 1/2$ ) for (5.39) to be achieved. It implies that the dividend must be large enough to be able to flip the increasing pattern to a decreasing one.

We also explored the behaviour of the model using plausible parameter values for the above case. As mentioned before,  $h$  and  $g$  are likely to have limited ranges. Thus, we can let  $h_{\min} = c_1 - c_2$  and  $h_{\max} = c_1 + c_2$  denote the possible minimum and maximum price of risk, respectively, and let  $g_{\min}$  and  $g_{\max}$  denote the possible

minimum and maximum dividend yield, respectively. The parameters  $\theta$ ,  $b_0$  and  $b_1$  can then be endogenously determined.

Table 5.1 summarises the parameter values applied in the implementation. Figure 5.2 shows how the price of risk varies over time with respect to the state of economy  $x$  for an investment horizon of 100 years. Figure 5.3 demonstrates the behaviour of the dividend yield with respect to the price of risk  $h$ , the state of economy  $x$ , and the stock index  $S$ , respectively.

## 5.6 Chapter Summary and Conclusions

There seems to be a consensus among financial economists that there is some predictability in stock index returns. It remains, however, something of a puzzle as to whether it is to do with pricing anomalies or whether it reflects the nature of the risk premia within the underlying economy.

The analysis in this chapter can be viewed as an attempt to approach this puzzle by setting up an equilibrium model of the asset price processes and showing that the time variation in asset expected returns (often postulated by some empiri-



cists) could be consistent with an equilibrium.<sup>8</sup> As demonstrated in the previous section, we have explored the possibility of a decreasing price of risk in the state of economy. It can be shown that this amounts to some degree of mean reversion in the expected returns and the agent displays a decreasing relative risk aversion. The model, therefore, indicates the potential that the resulting time-varying price of risk and time-varying risk aversion can better explain the time-varying equity risk premium.

Our assumptions of constancy of interest rate and volatility, which our time-homogeneous solutions were based on, are rather strong. It would seem natural to relax these assumptions and extend to models which can handle stochastic volatility and/or stochastic interest rate. Nevertheless, it is important and instructive to analyse the nature of the behaviour which is possible within this framework and

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<sup>8</sup>There is no difficulty in backing out the representative agent's utility. To see this, we only need to write down the definition for the intertemporal relative risk aversion as

$$f(S_t, t) = \frac{\alpha(S_t, t)}{\sigma} = -S_t \frac{J_{SS}(S_t, S_t, t)}{J_S(S_t, S_t, t)}. \quad (5.40)$$

In terms of  $x$ , it is given by

$$f(x_t, t) = 1 - \frac{J_{xx}(x_t, t)}{J_x} = -\frac{\mathcal{M}_x(x_t, t)}{\mathcal{M}(x_t, t)} = \frac{\alpha(x_t, t)}{\sigma}. \quad (5.41)$$

For any given time  $t \leq T$ , it is clear that (5.41) is simply an expression of ordinary differential equations. Thus, provided the functional form of the price of risk is known, we can integrate over  $x$  to recover the supporting utility  $J_x$ . Or equivalently, we can back out  $\mathcal{M}$  which, in turn, gives the state-price density function of the economy. More specifically, the last equality of (5.41) is equivalent to

$$\frac{\partial}{\partial x_t} \ln[\mathcal{M}(x_t, t)] = -\frac{\alpha(x_t, t)}{\sigma}. \quad (5.42)$$

Therefore, by integration, it follows that  $\mathcal{M}$  can be written as a function up to some constant  $A$ :

$$\mathcal{M}(x_t, t) = A \cdot \exp \left( -\frac{1}{\sigma} \int_{x_0}^{x_t} \alpha(\eta, t) d\eta \right).$$

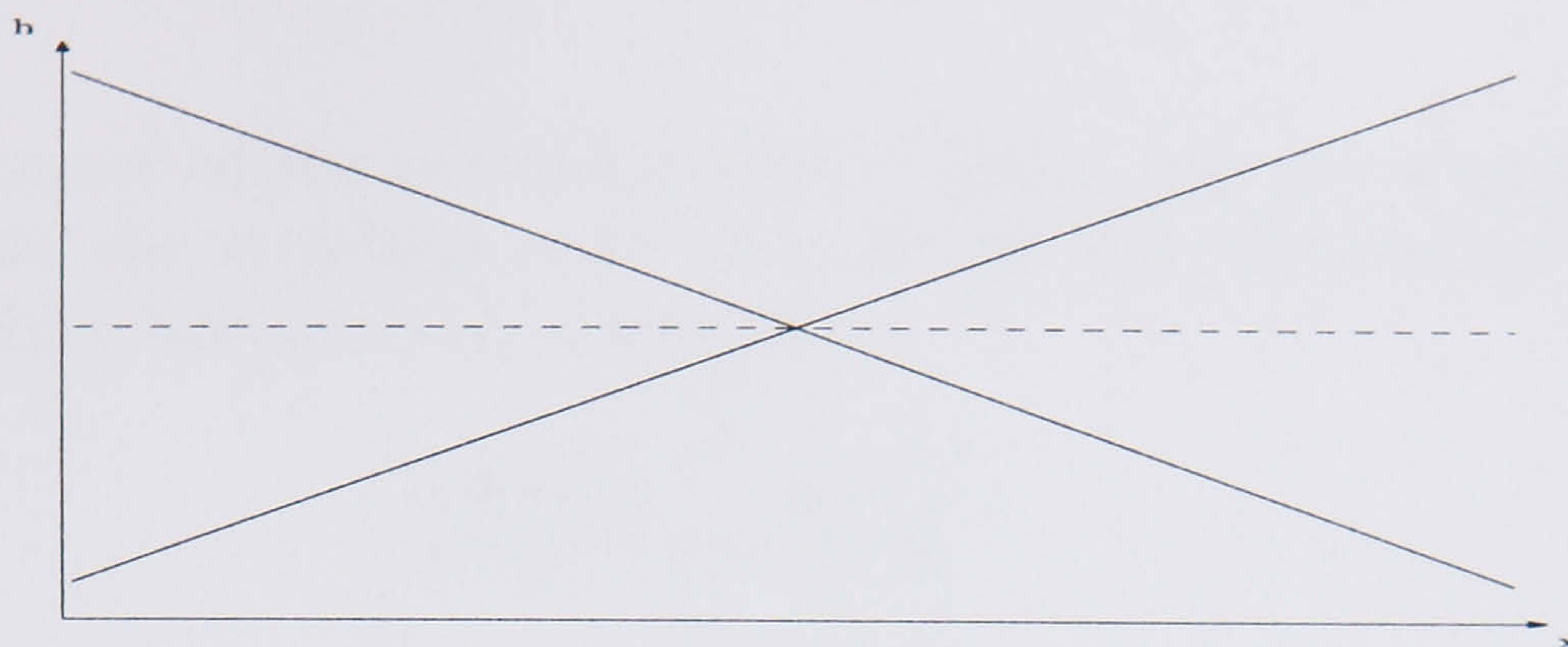
a representative agent equilibrium.

Although it has not been our focus in this chapter, it is worth noting that a careful examination of the consistency with the price process and the dividend process should be required when the intermediate consumption is included and a dividend process is postulated. This important issue is left for future research. We have also not intended to solve the so called *equity premium puzzle* of Mehra and Prescott [82]. It would be interesting to see how our model can be related to the puzzle. However, since we have assumed a time-additive utility, it seems less likely that this puzzle can be resolved in our framework.

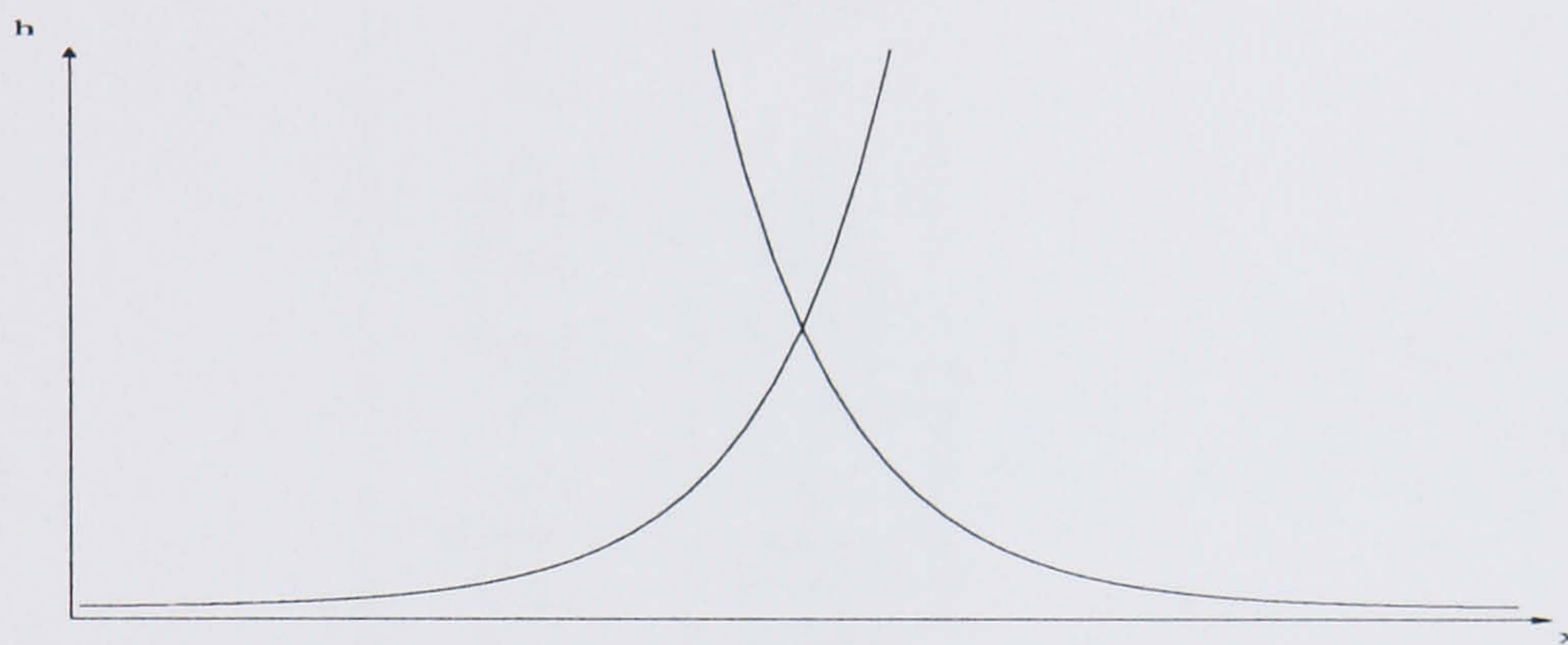
Finally, although the empirical issues are beyond the scope of our analysis, it would be nice to see how we can devise some kind of procedures so that the model can be empirically estimated/tested. These areas will challenge resourcefulness of future research.



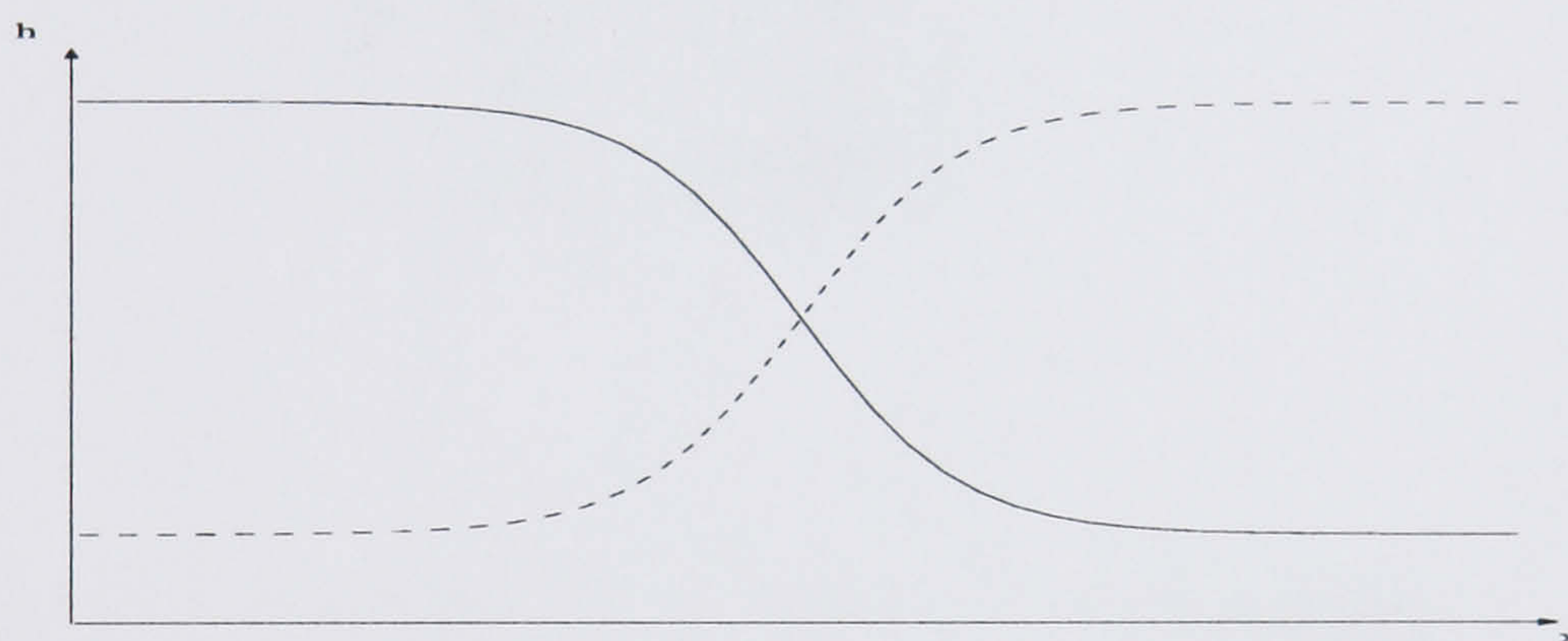
Panel I



Panel II



Panel III



Panel IV

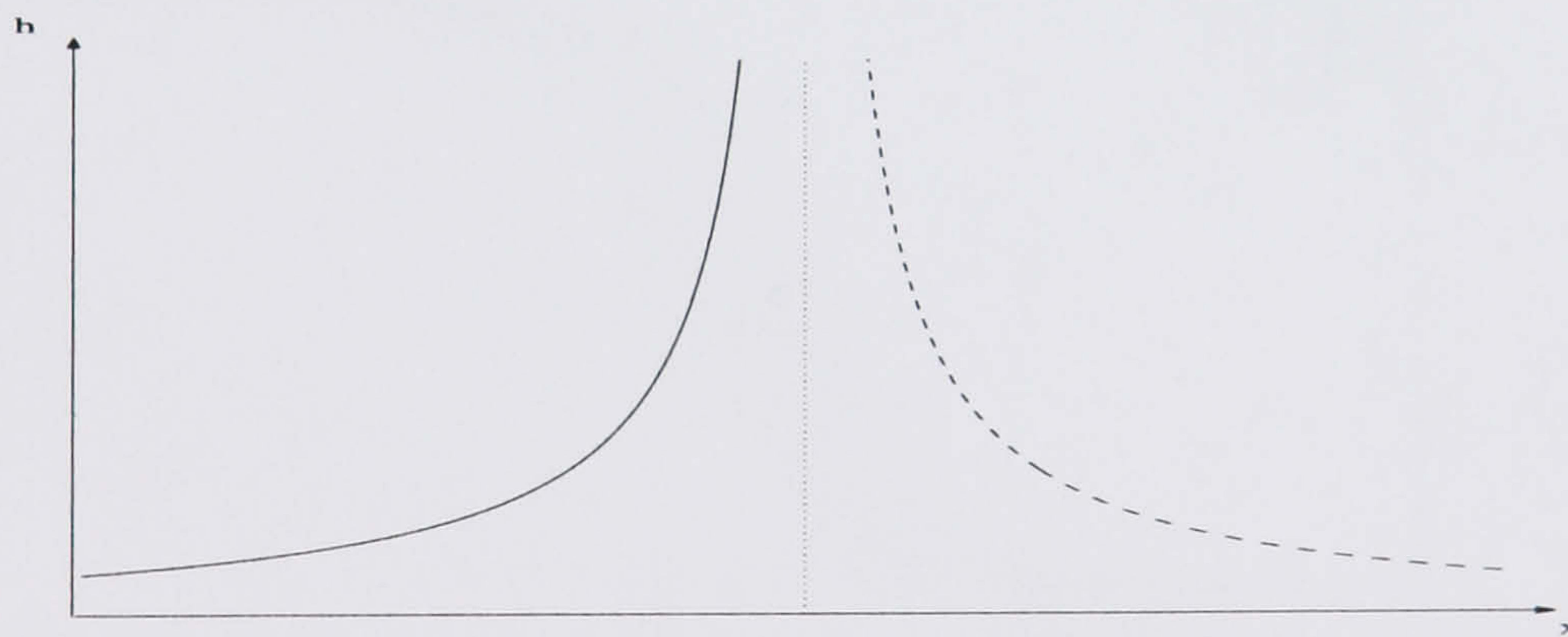


Figure 5.1: Behaviour of the price of risk: typical plots. These graphs are plotted based on the analytical solutions of the homogeneous case in Section 5.4. (The plots show linear, exponential, hyperbolic tangent, and hyperbolic forms, respectively.) The solid lines represent the patterns which can be obtained only from the economy with intermediate consumption, while the dashed lines represent the patterns which can also be achieved from the economy without intermediate consumption.



Table 5.1: Summary of the parameterisation. These parameter values are used to calculate the travelling wave price of risk and the corresponding dividend yield for a time horizon of 100 years.

Notation	Value
$r$	0
$\sigma$	0.15
$S_0$	1
$x_0$	0
$h_{\min}$	0.25
$h_{\max}$	0.75
$\theta$	-0.015
$k$	0
$g_{\min}$	0.015
$g_{\max}$	0.065
$p_0$	-0.01
$p_1$	0.6667

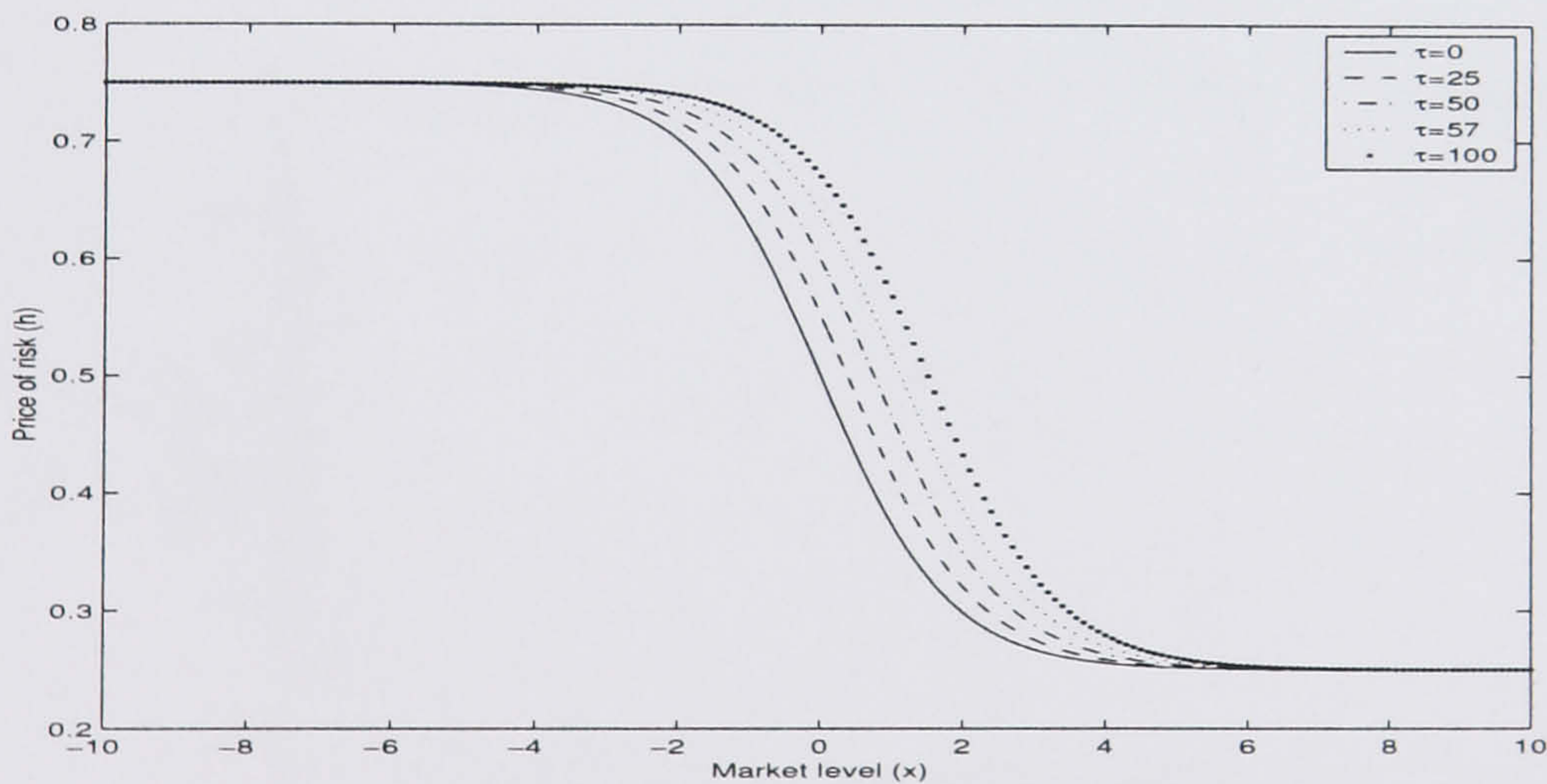
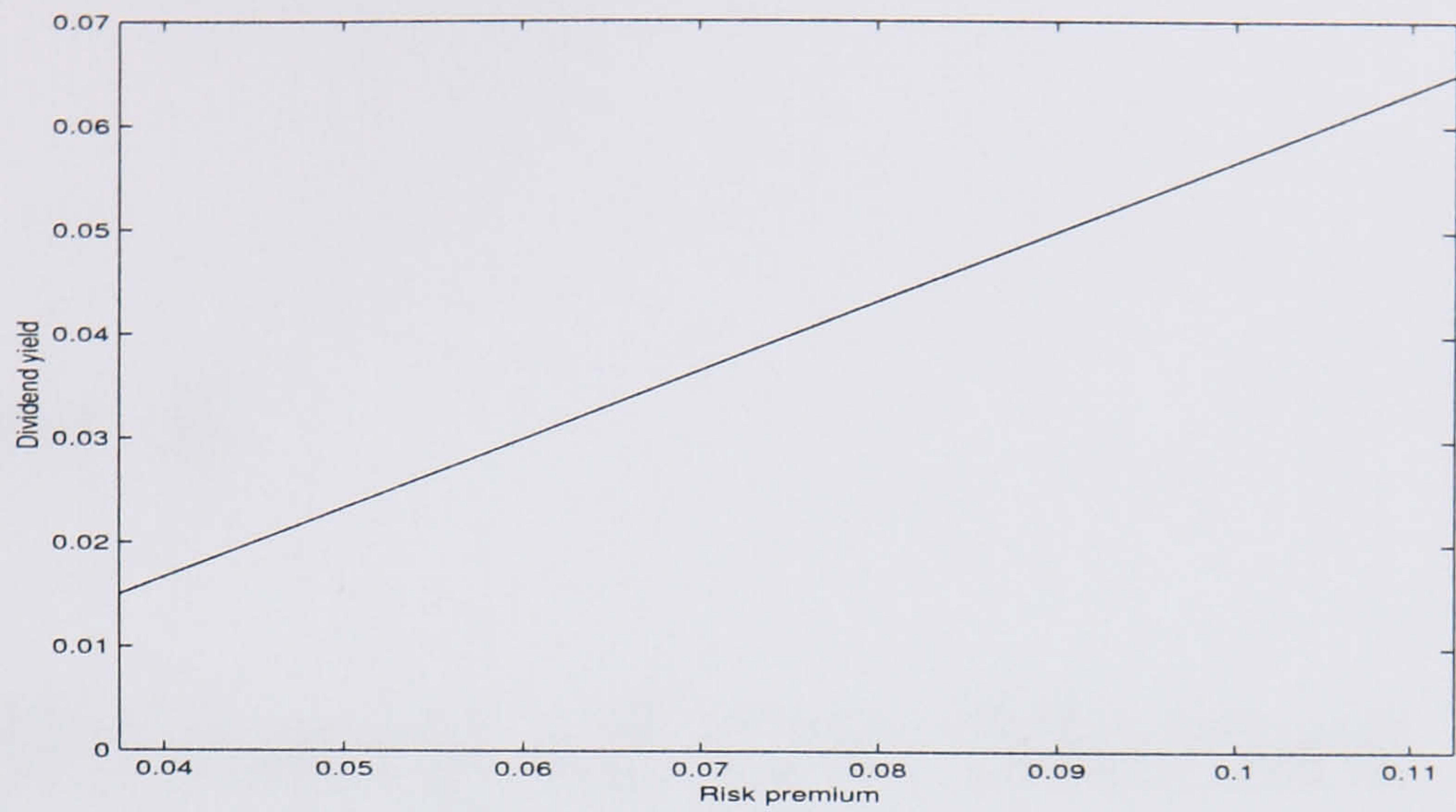


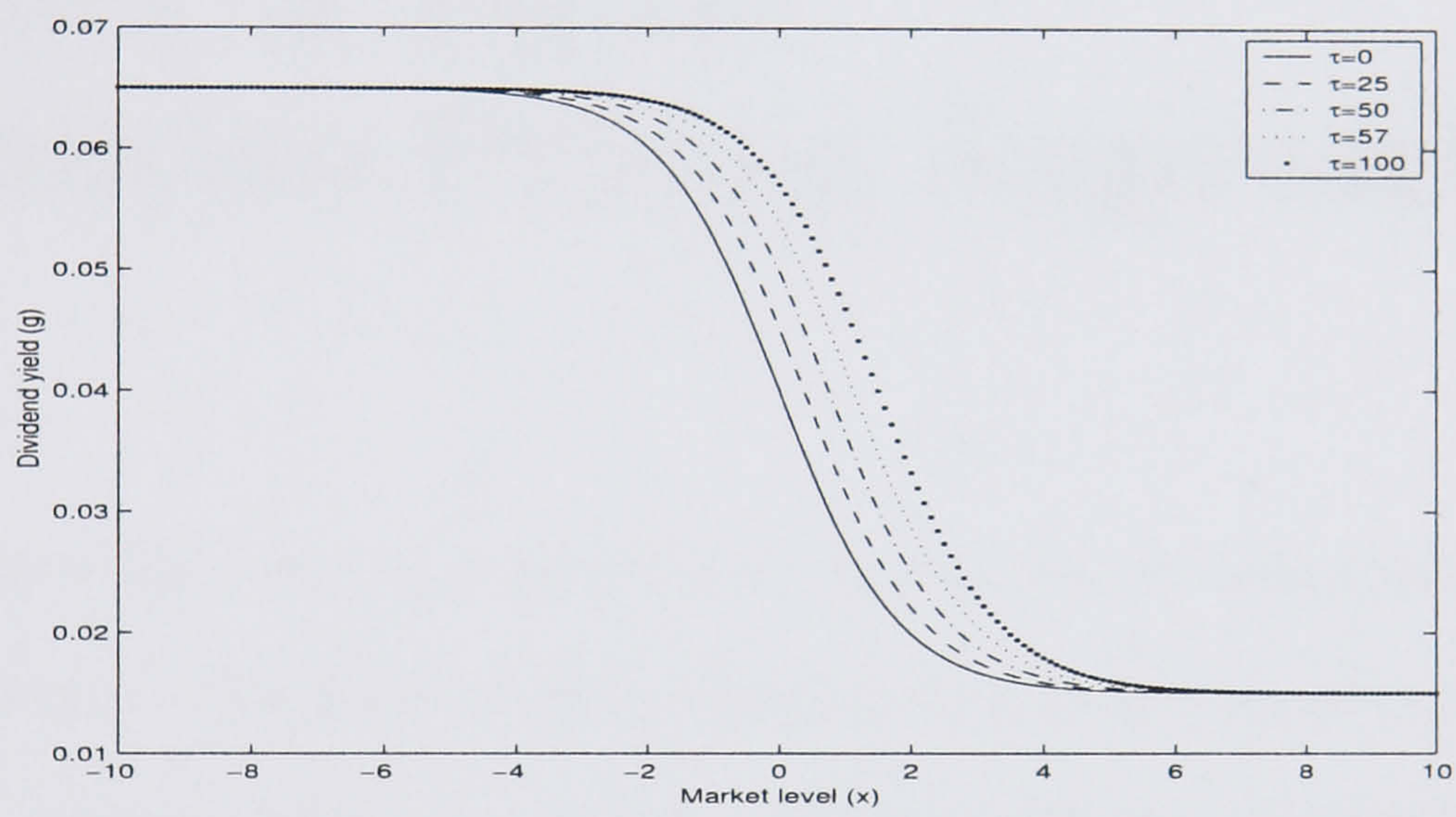
Figure 5.2: Price of risk: a travelling wave example. These plots are calculated from the travelling wave solution using the parameter values listed in Table 4.1. The total investment horizon is 100 years.  $\tau$  is the remaining time to terminal date.



Panel I



Panel II



Panel III

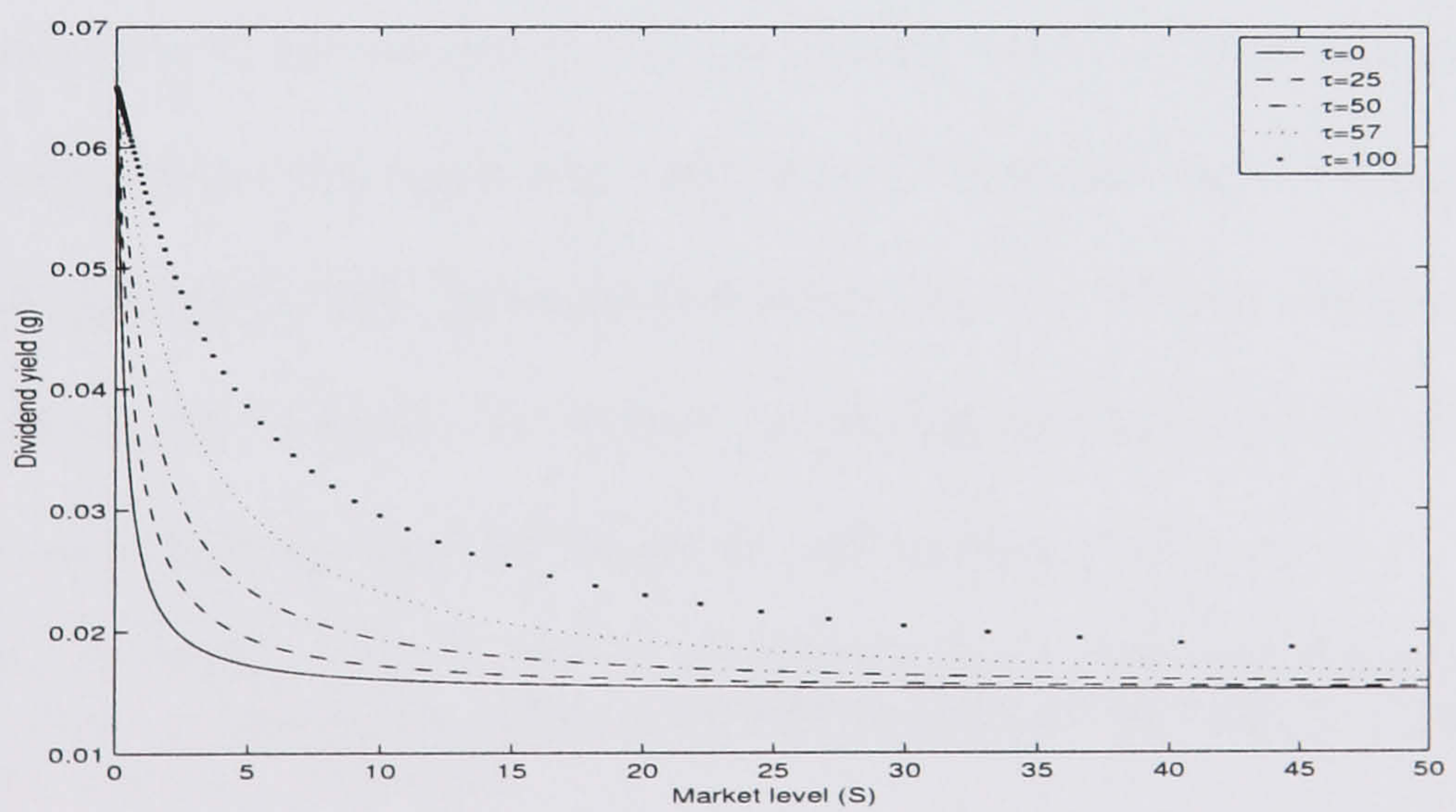


Figure 5.3: The dividend yield: An example. These plots are calculated based on the travelling wave solution for the price of risk and the corresponding dividend yield function using the parameter values listed in Table 4.1. The total investment horizon is 100 years.  $\tau$  is the remaining time to terminal date.



# Chapter 6

## The Efficiency of the Market Portfolio: A Payoff Distribution Pricing Approach

In this chapter and the next, we explore issues relating to the measurement of portfolio performance. The focus of this chapter is to examine numerically the efficiency of the market portfolio in a simple economy where no dividends and intermediate consumption are assumed. We postulate various forms of diminishing risk premium functions and apply the performance measure based on Dybvig's payoff distribution pricing model. Specifically, we construct a Monte Carlo simulation to explore the extent to which the market would be an inefficient investment, under alternative assumptions for the behaviour of the risk premium.<sup>1</sup>

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<sup>1</sup>This chapter is a modified version of an earlier working paper presented and published in the proceedings of the 1<sub>st</sub> Annual Conference of BDP, the Use of Derivatives in Portfolio Management, Porto, Portugal, May 1999.



## 6.1 Introduction

The market portfolio is an interesting research subject in financial economics. In particular, the risk premium of the market portfolio plays a significant role in the context of portfolio management. Despite substantial empirical interests and advances on this topic, there has been a lack of theoretical attention until recently to the way the market risk premium evolves through time. (See Hodges [60], p.592.) Some progress has been made through a handful of recent studies on the equilibrium asset price processes and now we are beginning to see a clearer picture of how the market risk premium would respond to the market level. (See for example, Bick [4, 5], He and Leland [57], Hodges and Carverhill [61] and Hodges and Selby [63].)

In the previous chapter, we proposed a continuous-consumption equilibrium model which is potentially capable of capturing a mean-reverting kind of behaviour in the market risk premium. Our interest in this chapter is, therefore, to ask a simple question: if a given price process of the market portfolio is not consistent with equilibrium, then how inefficient an investment it would become? The continuous-consumption model is by and large closer to the real word situation. However, it requires more complex equilibrium settings and for the purpose of evaluating the efficiency of the market portfolio, we will need to postulate both the price process and the dividend process.

For simplicity, in this chapter, we concentrate on the one-consumption case, i.e. an economy in which the stock is non-dividend-paying and the consumption

only occurs at time  $T$ . Our intention is to propose a framework that can be used to evaluate explicitly the possible efficiency loss when the market portfolio is indeed inefficient. The idea is as follows: First, from Hodges and Carverhill [61], we know that in a one-consumption Black-Scholes economy, the market risk premium must satisfy the Burgers' equation in order to be consistent with equilibrium. Hodges and Selby [63] further suggested that in this setting, a diminishing kind of behaviour cannot exist in a stationary one-consumption economy. This implies that when we postulate the premium function to be diminishing in the market index, it will not be consistent with equilibrium, and therefore the market index would become a stochastically dominated form of investment. Individuals might hold the market index at particular points in time, but not as a passive investment.

Based on the performance measure of Dybvig's [36, 37], we then construct a Monte Carlo simulation procedure to measure the magnitude of efficiency loss and explore the extent to which the market would be an inefficient investment, under alternative assumptions for the behaviour of the risk premium. Our simulation approach is different from Dybvig's original proposal in that we assume equal probable states under the risk neutral measure rather than the objective one. This approach enables us to uncover the density functions in an efficient manner. However, it requires some cautions at the stage of matching the optimal wealth and the correct state prices.

The remainder of this chapter is organised as follows. In Section 6.2. we set out the basic model and briefly review Dybvig's performance measure based on the payoff distribution pricing model (PDPM). We then describe the steps of our Monte



Carlo simulation scheme. In Section 6.3, we postulate a variety of diminishing risk premium functions and report the simulation results. Section 6.4 concludes the chapter.

## 6.2 Examining the Efficiency of the Market Portfolio

In this section, we start off by briefly recapitulating the basis of Dybvig's [36] payoff distribution pricing model using a discrete-time setting. We then provide the more convenient continuous-time representation for the similar concepts. Subsequently, the Monte Carlo simulation approach is constructed step by step for numerical evaluation of the market efficiency.

### 6.2.1 A Discrete-Time Illustration

Consider an economic agent whose optimisation problem is to maximise her expected utility over the wealth at the horizon date  $T$ , i.e.  $\mathbb{E}U(W_T)$ . When labelled by path  $\omega$ , the first order condition gives

$$U(W_{T\omega}) = \lambda e^{-rT} \frac{q_\omega}{p_\omega} \quad (6.1)$$

where  $\lambda$  is the Lagrange multiplier,  $q_\omega$  and  $p_\omega$  are the risk-neutral probability and the objective probability of reaching  $W_{T\omega}$ , respectively. Since the interest rate  $r$  is assumed to be constant, without loss of generality, we can refer to the ratio of  $q_\omega$  to  $p_\omega$  as the state-price density.

From (6.1), we obtain that  $W_T = U^{-1}(q_\omega/p_\omega)$ . By the assumption that the

marginal utility  $U$  is an increasing function, we can conclude that an efficient wealth function  $W_T$  must be decreasing in the state-price density. This means that the agent will purchase more wealth when it is cheap and will demand less when it is dear.

Recall that the market portfolio (or the market index) represents the wealth of the representative agent. Hence, an efficient market index  $S_T$  must have the same property, i.e. decreasing in the state-price density. As we have reviewed earlier, Dybvig cleverly exploited this property to form the payoff distribution pricing model and proposed a new performance measure. We will not repeat the machinery of the PDPM here. Instead, our purpose is to emphasise the idea that the knowledge of  $S_{T_\omega}$ ,  $p_\omega$  and  $q_\omega$  will be sufficient for us to measure the efficiency of  $S_T$ . The continuous-time representation is provided next.

### 6.2.2 The Continuous-Time Model

Consider an economy where there is a single risky asset (the stock or the market portfolio) and a riskless bond (the bank account). Let the price of the market portfolio  $S$  follow an Ito process of the form

$$\frac{dS}{S} = [r + \sigma\alpha(\cdot, t)]dt + \sigma dz, \quad (6.2)$$

where  $r$  and  $\sigma$  are constants and  $z$  is a Brownian motion under the objective probability measure  $\mathbb{P}$ . Let us denote by  $\mu$  the expected rate of return of  $S$  under  $\mathbb{P}$  assuming that it is a function of  $S$  and  $t$ . We then have  $\alpha$  in (6.2) as the adapted



process for the price of risk defined as  $\alpha \equiv \alpha(S, t) = (\mu(S, t) - r) / \sigma$ .<sup>2</sup> Assume that the market does not allow an arbitrage. This means that there exists a measure  $\mathbb{Q}$  such that it is equivalent to  $\mathbb{P}$  and such that the normalised price process (the bank account as the numeraire) is a martingale. Therefore, (6.2) can be rewritten as

$$\frac{dS}{S} = rdt + \sigma d\tilde{z}, \quad (6.3)$$

where  $\tilde{z}$  is a Brownian motion under the equivalent martingale measure  $\mathbb{Q}$ .

Note that our objective is to postulate the function  $\alpha$ . Therefore, the market index is lognormally distributed under  $\mathbb{Q}$  but will not be under  $\mathbb{P}$ . For this reason, we can not assume equal probable states like Dybvig's original proposal. Instead, we assume equal probability under  $\mathbb{Q}$ . Hence, it is more convenient to work with the change of measure  $\mathbb{P}/\mathbb{Q}$ . Denote by  $\Lambda_T$  this density ratio (i.e., the *Radon-Nikodym derivative*) and deduce that  $\Lambda$  follows an Ito process

$$\frac{d\Lambda}{\Lambda} = \alpha(S, t)d\tilde{z}, \quad \Lambda_0 = 1, \quad (6.4)$$

or

$$\Lambda_t = \exp \left( \int_0^t \alpha(s)d\tilde{z}_s - \frac{1}{2} \int_0^t \alpha^2(s)ds \right), \quad t \leq T. \quad (6.5)$$

Note that  $\Lambda_t$  is a  $\mathbb{Q}$ -martingale and if we take the bank account as numeraire, the reciprocal of  $\Lambda_T$  represents the marginal utility of the investor.

Similar to the approach we used in the previous chapter, we postulate  $\alpha$  as a

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<sup>2</sup>Assume that  $\alpha(S, t)$  satisfies Novikov's condition

$$\mathbb{E} \left[ \exp \left( \int_0^T \alpha^2(S, t)dt \right) \right] < \infty.$$

function of the log transformed market index and time. Specifically, we work with the new variable  $x$ , where

$$x_t = \ln(S_t) - \left(r - \frac{\sigma^2}{2}\right) t. \quad (6.6)$$

Note that  $x$  is a martingale under  $\mathbb{Q}$ -measure since

$$dx = \sigma d\tilde{z}. \quad (6.7)$$

### 6.2.3 The Simulation Approach

We can now describe the algorithm of our Monte Carlo simulation. The principle behind the simulations is straightforward. Firstly, we simulate by (6.7) the sample paths of  $x$  (which is the log transformation of  $S$ ) under  $\mathbb{Q}$ -measure. Then, we postulate the functional form for the price of risk  $\alpha$  (or equivalently the risk premium  $\mu - r$  since we have assumed constant  $r$  and  $\sigma$ ), and compute the change of measure  $\Lambda_T$  by (6.5). The objective probability  $\mathbb{P}$  can then be obtained through the transformation of measures. Finally, we sort (in the spirit of Dybvig [37]) to assign the efficient payoff in each state and then calculate the inefficiency cost for the market portfolio. We repeat the same procedure under alternative assumptions for  $\alpha$  in order to compare their implications for long term investment.

The details of each of the steps are set out as follows:

#### Step 1: Brownian Bridge Paths and Monte Carlo Simulation

Let  $n$  denote the number of subintervals such that  $0 = t_0 < t_1 < \dots < t_n = T$ .



Applying first-order Euler approximation for the discretisation of (6.7) yields<sup>3</sup>

$$\Delta X(t_i) = \sigma \Delta Z(t_i), \quad (6.8)$$

where  $\Delta t_i = t_{i+1} - t_i$  and  $\Delta Z(t_i) = Z(t_{i+1}) - Z(t_i)$  are normally distributed increments with mean zero and variance  $\Delta t_i$ . We can rewrite  $\Delta Z(t_i) = \sqrt{\Delta t_i} u$ , where  $u$  is an independent standard Gaussian distributed random number.<sup>4</sup>

Instead of drawing successive  $u$  from  $N(0, \Delta t)$  as in the crude Monte Carlo method, we propose to use the *Brownian bridge midpoint displacement* scheme to pin down the processes for the simulated paths. This means that we iteratively simulate the midpoint values from the two given endpoints and then map out to form one sample path.<sup>5</sup> Let  $M$  denote the number of paths to be simulated and let the whole horizon be divided by  $N$  bisections of equally spaced time steps. The time interval is then  $\Delta t = 1/2^N$ .

The Brownian bridge midpoint  $X_t$  from two given endpoints,  $X_{t_a} = a$  at  $t = t_a$  and  $X_{t_b} = b$  at  $t = t_b$ , is given by

$$X_t = \frac{1}{2}(a + b) + \frac{\sigma}{2}\sqrt{t_b - t_a} \cdot u, \quad \text{for } t_a \leq t = \frac{t_a + t_b}{2} \leq t_b. \quad (6.9)$$

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<sup>3</sup>We also experimented with the *second-order Milshtein approximation* as suggested in Kloeden, Platen and Schurz [74]. However, the results did not show significant improvements and therefore were omitted here.

<sup>4</sup>There are several numerical algorithms for generating these Gaussian variates. In our Monte Carlo simulation, we use a modified *Box-Muller* method which is computationally more efficient when a large amount of random numbers is to be generated.

<sup>5</sup>A description of the generalised Brownian bridge process is provided in Appendix A. By checking out moments of the distribution of a Brownian bridge process, one can verify that a Brownian bridge does produce a *Brownian motion*. More detailed exposition can be found in Bouleau and Lepingue [10].

The formula (6.9) is applied recursively in the following order of iteration

$$(0, T), \left(\frac{T}{2}\right), \left(\frac{T}{4}, \frac{3T}{4}\right), \left(\frac{T}{8}, \frac{3T}{8}, \frac{5T}{8}, \frac{7T}{8}\right), \dots$$

For example, at the first level, we have  $X_0 = \ln(S_0)$  and can easily generate  $M$  values of  $X_T$  by simulating  $M$  Gaussian random variables  $u_1$  so that  $X_T^j = X_0 + \sigma\sqrt{T}u_1^j$ , for  $j = 1, \dots, M$ . Then we move down to the second level to simulate  $M$  values of  $X_{T/2}$  from the endpoints  $X_0$  and  $X_T$  according to (6.9) so that for the  $j$ th path,

$$X_{\frac{T}{2}}^j = \frac{1}{2} \left( X_0 + X_T^j \right) + \frac{\sigma}{2} \sqrt{T} \cdot u_2^j.$$

At the third level, we then simulate  $X_{T/4}$  from  $(X_0, X_{T/2})$  and  $X_{3T/4}$  from  $(X_{T/2}, X_T)$ .

This procedure is repeated until the required level is reached.

## Step 2: The Change of Measure and the Objective Probability

Given a specified function  $\alpha$ , we can obtain the change of measure  $\Lambda_T$  by the discretised expression of (6.5)

$$\begin{aligned} \Lambda^{(n)}(T) = & \exp \left( \sum_{i=0}^{n-1} \alpha(X_{t_i}, t_i) \Delta Z(t_i) \right. \\ & \left. - \frac{1}{2} \sum_{i=0}^{n-1} \alpha^2(X_{t_i}, t_i) \Delta t_i \right). \end{aligned} \quad (6.10)$$

The transformation from the risk-neutral probability  $\mathbb{Q}$  to the objective probability  $\mathbb{P}$  is rather easy because in our simulations,  $d\mathbb{Q}$  is equal to  $1/M$  for every path. In principle, we shall expect the approximated exponential martingale  $\Lambda^{(n)}(T)$  converge to its continuous time counterpart  $\Lambda(T)$  as  $n$  approaches infinity. A criterion such as the following function can be employed to determine a suitable



$N$ ,

$$\delta^{(N)} = \frac{1}{\Lambda T} \sum_{j=1}^M |\Lambda^{(N)}(T) - \Lambda^{(N-1)}(T)|.$$

### Step 3: The dominating payoff and the inefficiency cost

The inefficiency cost of Dybvig comes from the difference between the asset price and the distributional price. Therefore, in order to calculate the distributional price of the market index with payoff  $S_T$  at the horizon  $T$ , we first need to allocate ‘correct’ payoff to each state so that it dominates the market payoff  $S_T$  in the sense that it has the same payoff distribution but has a cheaper price than  $S_0$ .

The allocation of dominating payoff can be achieved by some sorting and interpolating procedures. First, we sort on  $S_T$  and  $1/\Lambda_T$  respectively to obtain the cumulative relative frequency distributions  $F(1/\Lambda_T)$  and  $F(S_T)$  under the  $\mathbb{P}$  measure. Let us denote by  $p$  the relative frequency distribution of the states. We calculate a new cumulative relative frequency  $\rho$  associated with the sorted  $p$  such that  $\rho_1 \leq \rho_2 - \rho_1 \leq \rho_3 - \rho_2 \leq \dots \leq \rho_M - \rho_{M-1}$ . This allows us to interpolate from  $F(1/\Lambda_T)$  and  $F(S_T)$  to obtain a set of paired values of the change of measure and the corresponding dominating payoff, i.e.  $\left\{ \left( F_{1/\Lambda}^{-1}(\rho_j), F_S^{-1}(1 - \rho_j) \right) : j = 1, 2, \dots, M \right\}$ .<sup>6</sup> Finally, we can calculate the annualised percentage efficiency loss (APEL) of investing in the underlying market index:

$$\text{APEL} = \frac{1}{T} \left[ 1 - \frac{\sum_{j=1}^M F_S^{-1}(1 - \rho_j)}{\sum_{j=1}^M S_{T,j}} \right]. \quad (6.11)$$

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<sup>6</sup>In our examples, we apply cubic splines interpolation to obtain the corresponding efficient wealth levels. For the technique of spline fitting, see for example, Lancaster and Salkauskas [75] and Dierckx [34].

## 6.3 Implementation Results

In this section, we postulate the risk premium function and implement our simulation approach described before to see how inefficient the market portfolio would become under these assumptions. Our postulated risk premium functions are primarily inspired by the analytical results found in the earlier studies.

Specifically, recall from the work of Hodges and Carverhill [61], Hodges and Selby [63] and the analysis in Chapter 5 that in a Black-Scholes world without dividend and intermediate consumption, the market coefficients  $\mu$ ,  $r$  and  $\sigma$  behave in certain ways. In particular, the market price of risk  $\alpha$  must satisfy the Burgers' equation <sup>7</sup>

$$\alpha_\tau = \frac{1}{2}\sigma^2\alpha_{xx} + \sigma\alpha\alpha_x. \quad (6.12)$$

Our interest is on the diminishing form of risk premium function due to the reason that it seems to conform to empirical findings. Although Hodges and Carverhill [61] did provide a closed-form solution of  $\alpha$  that is decreasing in the market level  $x$ , it will collapse when one drifts backwards from the  $Z$ -shaped initial condition.<sup>8</sup> Therefore, we seek to find some other kind of functional forms that will be rather stationary. However, as we have reviewed in Chapter 5, a diminishing risk premium is not viable in a stationary Black-Scholes economy when there is no dividend and intermediate consumption. The only solution to the Burgers' equation in such

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<sup>7</sup>For a derivation of this equation, see Theorem 5.2 in Chapter 5.

<sup>8</sup>With the boundary condition  $\alpha(x, \tau) = \alpha(x, 0) = a - bx$  ( $a$  and  $b$  are constants), the closed-form solution to the PDE (6.12) is given by

$$\alpha = \frac{a - bx}{1 + b\sigma\tau}. \quad (6.13)$$



a stationary economy is that  $\alpha$  be either a constant or a travelling wave which increases at higher market levels. (See Hodges and Selby [63].)

Therefore, we assume some of the variants inspired by the travelling form solution. For convenience, we recall in the following the stationary solution as in Hodges and Selby [63]. Suppose the risk premium  $\alpha$  is time-homogeneous and is defined as

$$\alpha(x, \tau) \equiv y(u) \text{ where } u = x + \theta\tau \quad (6.14)$$

for some function  $y$  and some constant  $\theta$ . The two stationary solutions for function  $y$  are

$$y(u) = \begin{cases} c_1 = \frac{\theta}{\sigma} > 0 \\ \frac{(c_1+c_2)e^{s(u)}+(c_1-c_2)}{e^{s(u)}+1} & \text{for } y \in (c_1 - c_2, c_1 + c_2) \end{cases} \quad (6.15)$$

where  $s(u) = k + \frac{2c}{\sigma}u$ ,  $c_1$  and  $c_2$  are real ( $c_1 > c_2$ ), and  $k$  is a constant. Among these parameters,  $\theta$  in particular plays an important role as it determines the speed of the evolution. To summarise, we expect that the market portfolio will be efficient when the Burgers' equation is satisfied, e.g. (6.15), but will become an inefficient investment if not.

We now apply our approach to a number of examples. These examples are grouped into two cases. The first case considers an increasing risk premium with the same functional form as the travelling wave solution described above except that it has the 'wrong' speed (determined by  $\theta$  value). The second case considers a decreasing risk premium which is kind of a mirror image of the increasing case. We examine three scenarios with different speed parameters. The parameters used in each case are determined such that the price of risk  $\alpha$  is within the range of 0.4 and 0.7 and is equal to 0.55 at  $t = 40$ .



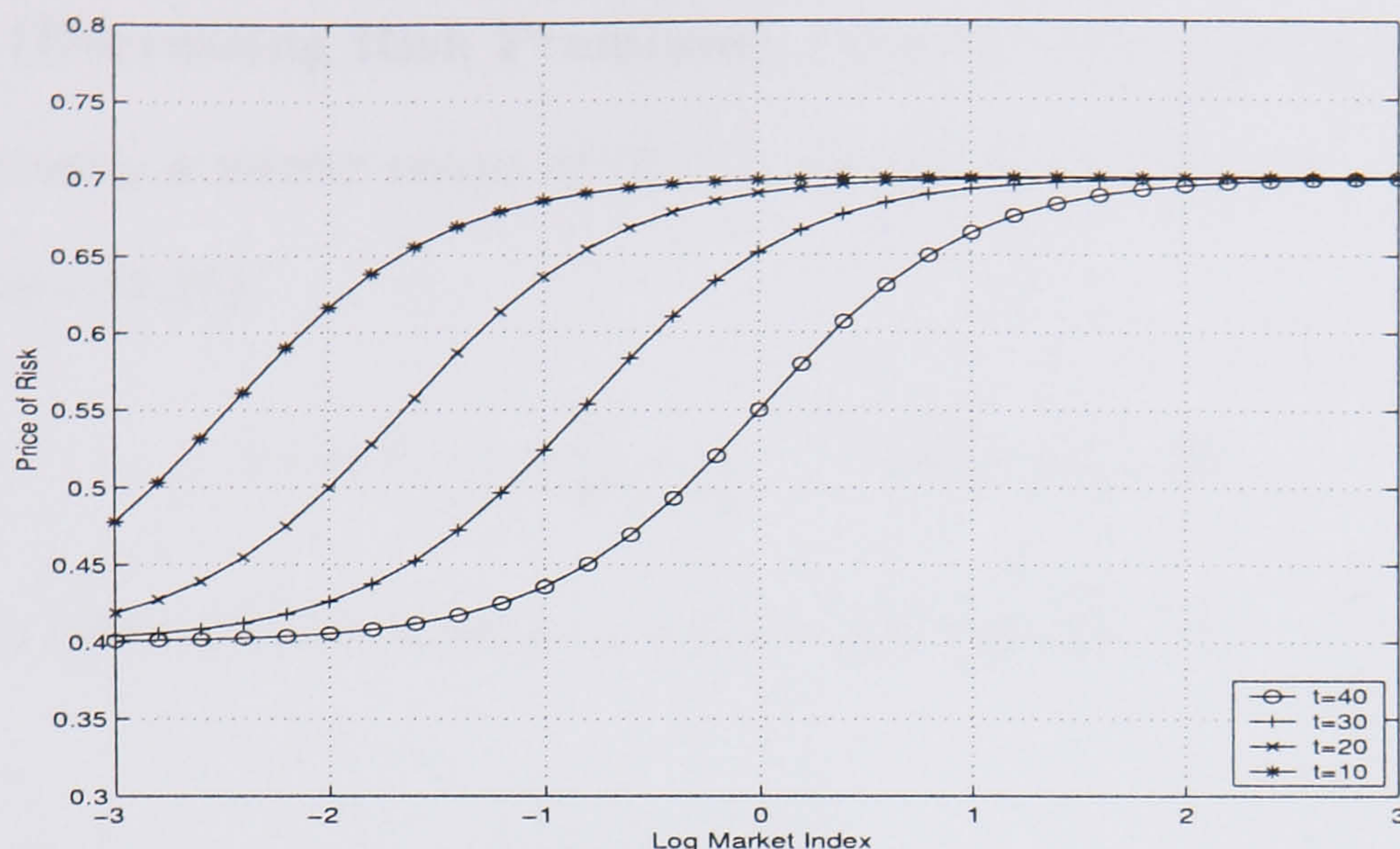


Figure 6.1: Increasing risk premium. This figure depicts the functional form (6.15) using parameter values  $b = 0.55$ ,  $c = 0.15$  and  $k = 0$ . When  $\theta = b\sigma > 0$ , the price of risk is the increasing function travelling to the right through time. When  $\theta = 0$ , the price of risk is represented by the fixed curve at the horizon date, i.e.  $t = 40$ .

For simplicity, we set out the parameter values as  $r = 0$ ,  $\sigma = 0.15$ . The number of bisections over the time horizon is  $N = 8$ . Table 6.1 shows the efficiency loss, across different lengths of investment horizons.

**Case 6.1 (Increasing Risk Premium)** Consider the functional form of (6.15) with parameter values  $b = 0.55$ ,  $c = 0.15$  and  $k = 0$ .

The behaviour of  $\alpha$  is shown in Figure 6.1. When  $\theta = b\sigma > 0$ , the market is efficient. However, when  $\theta = 0$  (as represented by the curve at  $t = 40$  (year) in Figure 6.1), the behaviour of  $\alpha$  is far from the evolution expected in equilibrium. Our simulations show that the inefficiency costs could range from 1.51 basis points per year over a ten-year investment to 17.03 basis points per year over a forty-year investment.



**Case 6.2 (Decreasing Risk Premium)** Consider a diminishing function which is approximately a mirror image of (6.15), similar to but not the same as the efficient solution (5.35):

$$\alpha = a + \frac{b}{e^{d(X+\theta\tau)} + 1}, \quad \text{where } b, d > 0. \quad (6.16)$$

Parameters values are set equal to  $a = 0.4, b = 0.3$  and  $d = 2$ .

The results for this case are reported as follows:

1. **Evolution to the right** ( $\theta = 0.0825$ ): The behaviour of  $\alpha$  is shown in Figure 6.2 (a). Since it behaves similarly to the one Hodges and Carverhill [61] have described for the decreasing form under which the Burgers' equation is satisfied, the inefficiency is expected to be mild. Our simulations show that it is nearly efficient.
2. **Evolution to the left** ( $\theta = -0.0825$ ): In order to compare with the previous case, we set the opposite speed, i.e.  $\theta = -0.0825$ . The behaviour of  $\alpha$  is shown in Figure 6.2 (b). Since it drifts away from the direction in which a solution of the Burgers' equation is supposed to evolve, we would expect that a bigger efficiency loss would occur. Our simulations show that the inefficiency costs range from 1.21 basis points per year over a ten-year horizon to 8.51 basis points per year over a forty-year horizon.
3. **Fixed form** ( $\theta = 0$ ): The curves at  $t = 40$  (year) in Figure 6.2(a) or Figure 6.2(b) represent this example. The inefficiency cost is 14.63 basis points per year over a forty-year horizon.



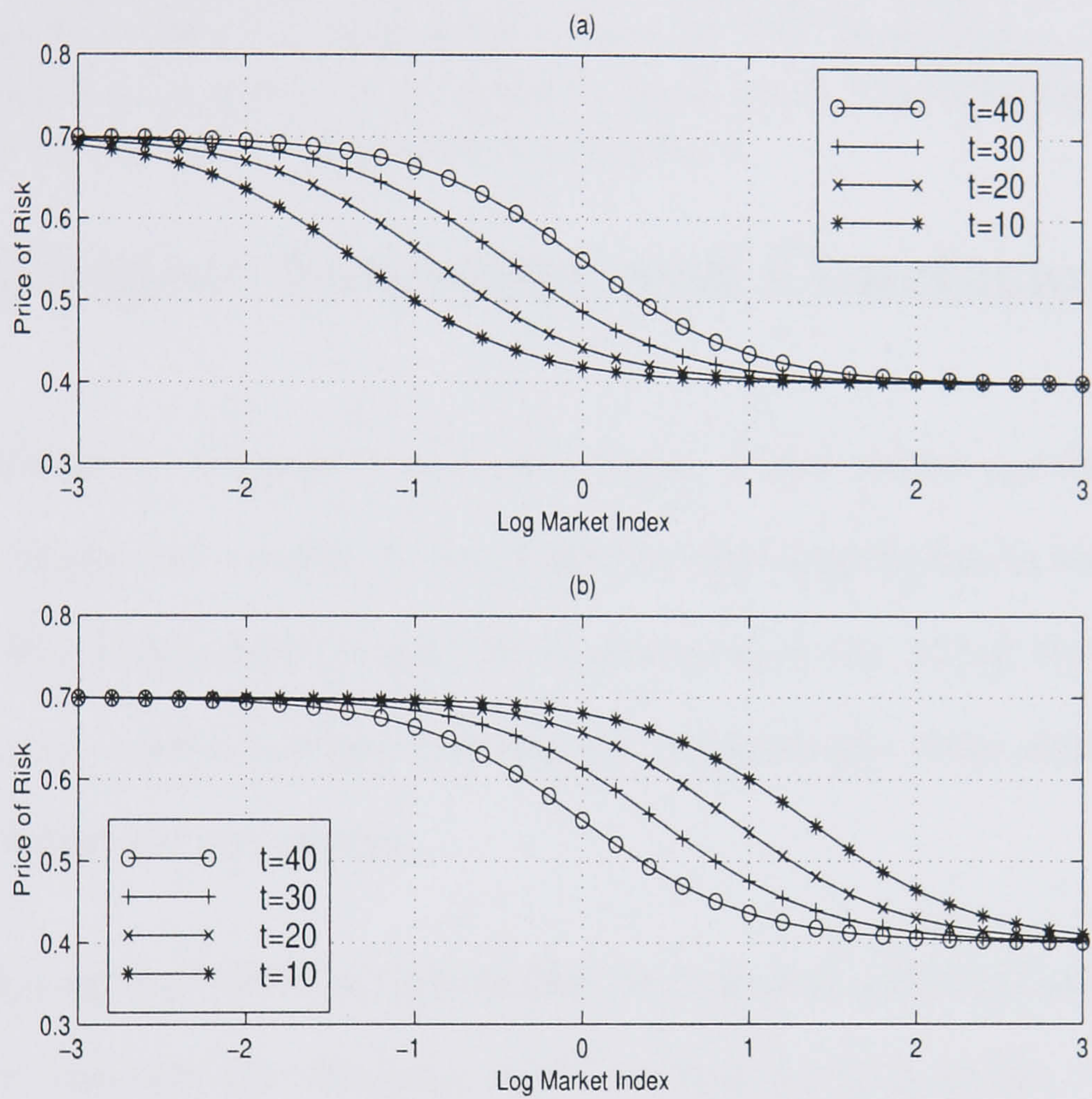


Figure 6.2: Decreasing risk premium. This figure depicts the functional form (6.16) using parameter values  $a = 0.4$ ,  $b = 0.3$ , and  $d = 2$ . Plot (a) shows the evolution to the right with  $\theta = 0.0825$ ; Plot (b) shows the evolution to the left with  $\theta = -0.0825$ . The case when  $\theta = 0$  is set equal to the curve at  $t = 40$  in either Plot (a) or Plot (b).



Efficiency Loss				
	T=10	T=20	T=30	T=40
A	1.51	4.83	11.35	17.03
B.1	0	0	0	0
B.2	1.21	3.29	6.20	8.51
B.3	0	0.33	4.53	14.63

Table 6.1: Efficiency loss of the market portfolio. Category A represents Case 6.1 and Category B represents Case 6.2. The simulations are repeated for alternative horizons from ten-year to forty-year. The efficiency losses are tabularised in terms of annualised basis points.

## 6.4 Chapter Summary and Conclusions

As demonstrated in Chapter 5, the price of risk of the market portfolio must behave in a certain way in order to be consistent with equilibrium in the underlying economy. When the specified asset price process does not satisfy the equilibrium conditions, the market portfolio will become stochastically dominated and, therefore, is an inefficient investment.

By applying the PDPM of Dybvig [36], we proposed a Monte Carlo simulation approach to examine the efficiency of the market portfolio under several postulated risk premium functions. Our numerical analysis confirmed the idea that a static market holding could be inefficient. However, the results suggest that the degree of inefficiency is relatively small in the situations we have investigated. The magnitude is of the scale of 20 basis points or less, even for a 40 year horizon.

There are few issues that still remain not completely explored. For example, it would be interesting to examine the economy with dividend and intermediate consumption such as the one we have studied in Chapter 5. We could estimate how

the dividend-price ratio changes empirically, and then see how close to efficient it might be. Some kind of dividend model such as the Lintner model [79] might also be useful. In the Lintner model, the instantaneous drift term of the dividend process is a function of earnings and dividend. Alternatively, we can postulate the dividend process such that the drift term is a function of market index and dividend.

Finally, we could extend our model to the case of stochastic interest rates, especially when we introduce the intermediate consumption. However, the case of stochastic volatility might not be a worthwhile extension.



# Chapter 7

## Portfolio Performance Measurement: An Expected Utility Based Approach

In this chapter we propose a new continuous-time performance measure based on intertemporal utility maximisation and explore the extent to which the new measure can be applied in the perspective of ex-post portfolio performance evaluation. A contingent claim estimation approach is employed to assign the nearest efficient values to the managed portfolio. We then implement the technique on some popular strategies through Monte Carlo simulation. The experimental results suggest that this technique is remarkably robust.

### 7.1 Introduction

Evaluating the performance of portfolio managers has generated a substantial amount of interests in the financial economics literature over the last three decades.

The growing popularity of active asset management through professionally managed portfolios also highlights the importance of such evaluations to be made available to investors. However, although the principle behind performance evaluation is rather straightforward, the problems involved are in general quite difficult.

In practice, fund managers usually employ some sort of models to forecast returns of market indices and individual stocks and utilise this information for timing the market and/or picking stocks. Therefore, in principle, a successful performance evaluation procedure should at least be able to answer two questions: (1) has the fund manager added values in providing his managerial skill? (2) how was the fund's performance achieved - through market timing, stock selection or other sources? As reviewed in Chapter 2, although there have been various performance measures which aim to answer the above questions, they are mostly conducted in the CAPM framework. Moreover, the attempt on performance decomposition is theoretically difficult and often results in erroneous conclusions.

Due to above reasons, our objective in this chapter is on assessing the efficiency of the dynamically managed portfolio strategies (i.e. adjusting the asset mix over time according to some rules) given return forecasts. In other words, we are not concerned with whether the manager possesses any forecasting ability. As we shall see later, for simplicity, we deliberately assume that the expected return is constant.

As demonstrated by Dybvig [37], a number of dynamic portfolio strategies followed by practitioners are significantly inefficient. The inefficiencies arise not because of imperfect diversification across stocks, but because of poor diversifica-



tion over time. (see Hodges [59].) For example, a manager who follows a market timing strategy but has actually no forecasting ability will be throwing investors' money away. Similarly, other popular strategies such as stop-loss, lock-in and repeated portfolio insurance were also shown, by Dybvig, to be inefficient. The amount of inefficiency costs could be substantial.

A natural question is then to ask how the performance of a dynamically managed portfolio can be measured, assuming that the manager has no forecasting ability, or, to put it another way, what is an appropriate yardstick to use when evaluating the performance of a portfolio in which the asset mix varies through time. Conceptually, the manager should aim to follow some kind of efficient strategy and the performance can be measured on the basis of how the managed portfolio deviates from the benchmark (i.e. the efficient strategy). However, the performance measured used for this evaluation need to be properly defined.

In this respect, traditional performance measures proposed in the literature do not provide the answer. As we have reviewed in Chapter 2, it is now well recognised that traditional CAPM based reward-to-variability measures such as Sharpe ratio [96] and Jensen's alpha [66] are inadequate for evaluating the performance of portfolios with non-linear payoffs. The possibly very distorted payoff distribution from some options strategies or dynamic trading is the major reason why a mean-variance framework cannot be used in comparing performance. Unfortunately, as pointed out in Glosten and Jagannathan [48], although the invalidity of mean-variance based measures is widely documented, there has not been many explicit operational procedures proposed for practical applications. There-

fore, the purpose of this chapter is to take on the challenge of the distributional problem as well as to develop a general methodology for evaluating performance of dynamically managed portfolios.

Our work is motivated by Dybvig's payoff distribution pricing approach, which has been reviewed and applied in earlier chapters. The performance measure based on his approach provides an exact solution for quantifying the global inefficiency of a predefined strategy (i.e. the strategy is known). More specifically, Dybvig evaluates the *ex-ante* performance of a strategy by defining the inefficiency cost as the difference between the asset price of the portfolio and the cheapest possible cost of generating the same payoff distribution. That is, Dybvig's inefficiency cost can be expressed as

$$w_0 - w_0^* = \mathbb{E}[\xi_T(W_T - W_T^*)] \quad (7.1)$$

where  $w_0$  and  $W_T$  are values of the portfolio strategy at time 0 and T, respectively, and  $w_0^*$  and  $W_0^*$  are values of the corresponding cheapest cost strategy.

In the *ex-post* analysis, however, it is not clear how his approach can be applied since we only observe a single realisation (say path  $i$ ) of the portfolio's values (or at most the portfolio's risk exposures). It can be seen that although  $w_0 - w_0^*$  in (7.1) will always be nonnegative,  $W_{T,i} - W_{T,i}^*$  will not. Thus, an important question is how the inefficiency cost should be distributed (or assigned) across all the possible paths. In our analysis, we argue that in the intertemporal setting the efficiency loss can be interpreted as the present value of the losses of certainty equivalent wealth over the investor's investment horizon. Any departure from the associated optimal strategy (or the efficient strategy) will result in an efficiency loss. This



approach is particularly useful when it comes to ex-post evaluation since it enables us to assign a nonnegative inefficiency cost to a single realisation.

In principle, an optimal strategy can be found by maximising the investor's expected utility, provided that the market return generating processes and the investor's preferences are known. However, in practice, it is often improbable to obtain an accurate information on the investor's preferences, especially for hedge funds. Therefore, in evaluating the performance of a realised wealth path of a portfolio whose actual strategy is unknown, it is necessary to infer the associated optimal strategy from the available observations.

Our approach to estimating the efficient strategy is very similar to that of Hodges [59] and is closely related to that of Glosten and Jagannathan [48]. In our analysis, the value of the managed portfolio is approximated by the value of a portfolio of options written on some reference index. Since the estimated values are supposed to be the closest to the portfolio values, we are actually evaluating the performance of the portfolio manager in a most charitable way. As a result, some strategies are more likely to be exposed as inefficient than others, depending on the nature of the strategies.

The rest of the chapter is organised as follows. Section 7.2 sets out the derivation of the new performance measure in a continuous-time setting. The estimation procedure for ex-post evaluation is described in Section 7.3. Section 7.4 then implements the technique on some popular strategies such as the stop-loss rule and the lock-in strategy to investigate how the ex-post cost is distributed. A discussion of how the model can be generalised and extended is provided in Section 7.5.

Section 7.6 finally concludes the chapter.

## 7.2 The Expected Utility Based Performance Measure

Consider a complete market and an intertemporal setting. As we have discussed before, an efficient strategy refers to a portfolio strategy that maximises the investor's expected utility. In principle, the efficient strategy can be obtained by solving the investor's optimal portfolio choice problem, provided that the investor's preferences are known. If the investor's preferences are unknown, it is in general quite difficult to see whether a particular strategy is optimal to the investor or not. In this analysis, we use the notion of certainty equivalence and define the total period inefficiency cost as the (annualised) expected value of the accumulated discounted loss of certainty equivalent wealth over the investment horizon. Suppose the horizon is  $H$  years. Then the above opportunity cost, denoted by  $\Psi^H$ , can be expressed as

$$\Psi^H = \frac{1}{H} \mathbb{E}_0^Q \left[ \int_0^H e^{-rt} \psi(t, \omega) dt \right], \quad (7.2)$$

where  $\psi$  denotes the instantaneous loss of certainty equivalent wealth associated with the realised strategy. Note that the expectation is taken under the risk neutral probability.

The task of quantifying the instantaneous loss  $\psi$  can be achieved by solving the intertemporal optimal portfolio choice problem and analysing the welfare effect arising from employing a suboptimal portfolio strategy other than the optimal one.



Now consider a standard portfolio choice problem. Assume that there is a risk averse investor who aims to maximise his or her expected utility of terminal wealth. There are two financial assets available for costless and continuous trading. Assume that the price of the risky asset  $S$  follows a geometric Brownian motion with a constant risk premium and the price of the locally riskless asset  $B$  grows at the constant riskless rate  $r$ . That is,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dz_t, \quad (7.3)$$

$$\frac{dB_t}{B_t} = r dt \quad (7.4)$$

where  $\mu$ ,  $\sigma$  and  $r$  are constant and  $z_t$  is a standard Brownian motion under the investor's subjective probability.

The optimisation problem is then given by

$$\max \mathbb{E}_0[U(W_T)] \quad (7.5)$$

subject to

$$w_0 = \mathbb{E}_0[\xi_T W_T] \geq 0, \quad (7.6)$$

$$dW_t = [W_t r + \phi_t(\mu - r)] dt + \phi_t \sigma dz \quad (7.7)$$

where  $W$  is the wealth level and  $\phi$  (the control variable) is the amount of money invested in the risky asset.

As is standard in the literature, the indirect utility is defined as

$$J(W_t, t) = \max \mathbb{E}_t[U(W_T)], \text{ where } J(W_T, T) = U(W_T),$$

and satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$J(W_t, t) = \max \mathbb{E}_t[J(W_{t+dt}, t + dt)]. \quad (7.8)$$

It is well known that the solution to this optimisation problem is that

$$\phi^*(W, t) = \frac{\mu - r}{\sigma^2} \left( -\frac{J_W}{J_{WW}} \right) \quad (7.9)$$

which will in turn make  $J(W, t)$  in (7.8) a martingale:

$$J(W_t, t) = \mathbb{E}_t[J(W_{t+dt}^*, t + dt)], \quad (7.10)$$

where  $W_{t+dt}^*$  is the wealth at time  $t + dt$ , achieved by following the time- $t$  optimal strategy  $\phi_t^*$ .

Now consider a strategy which at time  $t$  invests  $\phi_t$  dollars in the risky asset.  $\phi_t$  can be decomposed into two components:  $\phi_t^*$  and  $\epsilon_t$ , where  $\phi_t^*$  is the (universally) efficient strategy derived from (7.9) and  $\epsilon_t$  is the residual term. We further assume that the manager has no forecasting ability (i.e., no market timing ability) and aims to follow the efficient strategy. Therefore, it is convenient to assume that  $\epsilon_t$  follows a normal distribution  $\mathcal{N}(0, \nu^2)$  and is orthogonal to  $\phi_t^*$  and the innovation  $dz_t$ . Namely,

$$\phi_t = \phi_t^* + \epsilon_t, \quad (7.11)$$

$$\mathbb{E}[\epsilon_t] = 0, \quad (7.12)$$

$$\text{var}[\epsilon_t] = \nu^2, \quad (7.13)$$

$$\mathbb{E}[\phi_t^* \epsilon_t] = 0, \quad (7.14)$$

$$\mathbb{E}[\phi_t^* dz_t] = 0. \quad (7.15)$$



Thus, the instantaneous change of wealth will be given by

$$\begin{aligned}
dW_t &= [W_t r + \phi_t(\mu - r)] dt + \phi_t \sigma dz_t \\
&= [W_t r + (\phi_t^* + \epsilon_t)(\mu - r)] dt + (\phi_t^* + \epsilon_t) \sigma dz_t \\
&= dW_t^* + \epsilon_t [(\mu - r)dt + \sigma dz_t]
\end{aligned} \tag{7.16}$$

where  $dW_t^*$  denotes the instantaneous increment of wealth generated by the strategy  $\phi_t^*$ . It shows that the deviation from the optimal increment of wealth can be decomposed into the *extra* risk premium and the stochastic element caused by  $\epsilon_t$ .

Consequently, we can derive the indirect utility for the next instant from  $t$  as follows:

$$\begin{aligned}
&J(W_{t+dt}, t + dt) \\
&= J(W_t, t) + J_t dt + J_W(dW_t) + \frac{1}{2} J_{WW}(dW_t)^2 \\
&= J(W_t, t) + J_t dt + J_W [(dW_t^*) + \epsilon_t(\mu - r)dt + \epsilon_t \sigma dz_t] \\
&\quad + \frac{1}{2} J_{WW} [(dW_t^*)^2 + \epsilon_t^2 \sigma^2 dt + 2\phi_t^* \epsilon_t \sigma^2 dt] \\
&= J(W_{t+dt}^*, t + dt) + J_W \epsilon_t \sigma dz_t \\
&\quad + \left[ J_W \epsilon_t (\mu - r) + \frac{1}{2} J_{WW} (\epsilon_t^2 \sigma^2 + 2\phi_t^* \epsilon_t \sigma^2) \right] dt.
\end{aligned} \tag{7.17}$$

Substituting (7.9) into (7.17) yields

$$\begin{aligned}
J(W_{t+dt}, t + dt) &= J(W_{t+dt}^*, t + dt) \\
&\quad + J_W \epsilon_t \sigma dz_t + \frac{1}{2} J_{WW} \epsilon_t^2 \sigma^2 dt,
\end{aligned} \tag{7.18}$$

or more intuitively,

$$\begin{aligned} J(W_{t+dt}, t+dt) &= J(W_{t+dt}^*, t+dt) \\ &\quad + J_W \left[ \epsilon_t \sigma dz_t - \frac{1}{2} \gamma_t \epsilon_t^2 \sigma^2 dt \right]. \end{aligned} \quad (7.19)$$

where  $\gamma_t = -J_{WW}/J_W$  is the absolute risk aversion at  $t$ .

By inspection of (7.19), it can be seen that the deviation between the indirect utility from strategy  $\phi$  and the indirect utility from  $\phi^*$  is attributed to two effects: the differential utility caused by the stochastic element of incremental wealth attributed to  $\epsilon_t$  and the *cost* of not following the optimal strategy. The more risk averse the investor, the higher the cost. It is also important to point out that while this cost function ensures a negative effect on the indirect utility when a suboptimal strategy was implemented, it could be offset or deepened further purely by chance, depending on the comovement of  $\epsilon_t$  and  $dz_t$ .

However, under the assumption that the manager has no market timing ability,<sup>1</sup> we can take conditional expectation on both sides of (7.19) and apply the **martingale** result (7.10) to obtain the **supermartingale** expression for the indirect utility resulting from the **suboptimal strategy**  $\phi$ :

$$\mathbb{E}_t[J(W_{t+dt}, t+dt)] = J(W_t, t) - \alpha_t dt, \quad (7.20)$$

where

$$\alpha_t = -\frac{1}{2} J_{WW}(W_t, t) \epsilon_t^2 \sigma^2.$$

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<sup>1</sup>This is assured by our assumption that  $\mu$  is a constant. The relaxation of this assumption will be discussed later.



Assume that

$$J(\hat{W}_t, t) = \mathbb{E}_t[J(W_{t+dt}, t + dt)]. \quad (7.21)$$

It then follows that the instantaneous loss of certainty equivalent wealth, conditional on  $W_t$  and  $\epsilon_t$ , can be given approximately by

$$\psi_t|W_t, \epsilon_t = \left( \hat{W}_t - W_t \right) |W_t, \epsilon \approx \frac{\alpha_t}{J_W} = \frac{1}{2} \gamma_t \epsilon_t^2 \sigma^2. \quad (7.22)$$

The partial derivatives in (7.22) can be substituted out by applying the equality (7.9). That is,

$$\psi_t|W_t, \epsilon_t \approx \frac{\mu - r}{2} \frac{\epsilon_t^2}{\phi_t^*}. \quad (7.23)$$

Rewriting (7.23) and then substituting into (7.2) result in our approximate expression of measure of inefficiency cost:

$$\Psi^H \approx \frac{\mu - r}{2H} \mathbb{E}_0^Q \left[ \int_0^H e^{-rt} \frac{\epsilon_t^2}{\phi_t - \epsilon_t} dt \right]. \quad (7.24)$$

The annualised percentage efficiency loss (APEL) is simply equal to  $\Psi^H/W_0$ .

### 7.3 A Contingent Claim Approach to Ex-Post Performance Evaluation of Portfolio Strategies

In this section, we set out the mechanics of the contingent claim approach to estimating the optimal strategy (or the nearest efficient strategy). First we discretise to obtain a pathwise performance measure based on the continuous-time result. We then describe in detail the estimation procedure for the purpose of ex-post evaluation.

### 7.3.1 The Ex-Post Performance Measure

Consider a portfolio with initial value  $W_0$ . Suppose that the intended investment horizon was  $H$ , but we could only observe the portfolio values  $\{W_0, W_1, W_2, \dots, W_n\}$  at the limited time points  $\{t_0, t_1, t_2, \dots, t_n\}$ , where  $t_0 = 0$  and  $t_n = T$ . Moreover, assume that the time intervals are equal (i.e.,  $t_{i+1} - t_i = \Delta t$  for  $i = 0, \dots, n-1$ ) and  $T \leq H$ . The strategy was revealed through time in terms of the amounts invested in the risky asset, i.e.,  $\{\phi_0, \phi_1, \phi_2, \dots, \phi_{n-1}\}$ .

We can rewrite the discrete time analogy of (7.24) to obtain the pathwise and truncated APEL as

$$APEL = \frac{\mu - r}{2TW_0} e^{-r\Delta t} \sum_{i=0}^{n-1} \frac{\hat{\epsilon}_i^2}{\hat{\phi}_i - \hat{\epsilon}_i} \Delta t \quad (7.25)$$

where  $\hat{\epsilon}_i$  is the difference between the realised strategy  $\hat{\phi}_i$  and the nearest efficient strategy  $\hat{\phi}_i^*$  at the discrete point in time  $t_i$ .<sup>2</sup>

### 7.3.2 The Estimation Procedure

To uncover what the manager actually did, we need to estimate the nearest efficient portfolio. Therefore, we need to find a convenient basis for the monotonically increasing payoffs of the nearest efficient strategy to be represented as a function of the underlying value (i.e. the stock index). One simple way to achieve this is to form a simple linear approximation. More specifically, we first divide the space of the terminal stock price into a number of subintervals. Then for each subinterval, a

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<sup>2</sup>Since the integral of equation (7.24) is Riemann integrable, by subdividing  $[t_0, t_n]$  into finer subintervals, (7.25) would converge to (7.24).



straight line with nonnegative slope will serve as the basis function. Therefore, by combining these straight lines with nonnegative amount, we have formed a globally continuous, non-decreasing and piecewise linear basis function for estimating the nearest efficient function. This basis function also has an option interpretation. The non-decreasing straight line in each subinterval can be interpreted as the payoff of a bull spread (except the two far end of subintervals) with two end values of the subinterval as the strike prices.

The above procedure is explained further as follows. The space of the date- $H$  stock price is divided into  $J + 1$  intervals. Thus, We have a set of different values  $0 < K_1 < K_2 < \dots < K_J < \infty$  that  $S_H$  might attain at the horizon  $H$ . We then define the following  $J + 1$  options, whose values are denoted by  $C_1, C_2, \dots, C_{J+1}$ , based on the value of  $S_H$ .  $C_1$  is a short put option with the strike price of  $K_1$ .  $C_2, \dots, C_J$  are call spreads defined for each associated interval. For example,  $C_2$  is a call spread defined within the interval  $[K_1, K_2]$  and its payoff is: 0 for  $S_H < K_1$ ,  $S_H - K_1$  for  $S_H \in [K_1, K_2]$ , and  $K_2 - K_1$  for  $S_H > K_2$ . Finally,  $C_{J+1}$  is a call option with the strike price of  $K_J$ .

At each point in time, the values of these options can be calculated according to the option pricing formula. Note that a call spread can be decomposed to long a call with a strike  $K_L$  plus short a call with a strike  $K_H$ , for  $K_L < K_H$ :

Valuing the payoff function of the nearest efficient strategy can then be viewed as valuing a linear combination of a riskless investment and the portfolio of options. Thus, given the assumption that during the assessment period, the weights are constant, the value of the nearest efficient portfolio at each time point  $t$  is given

by

$$C(S, t; H) = B(t, H) + \sum_{j=1}^{J+1} w_j C_j(S_H) \quad (7.26)$$

where  $B(t, H) = e^{-r(H-t)} B(H, H)$  and  $B(H, H)$  is equal to a positive constant that represents a riskless payoff at time  $H$ .

Let  $x_{t,j}(S_t)$  denote the amount of money to be invested in the risky asset at date  $t$  ( $t = 0, 1, \dots, T$ ) in order to obtain the contingent claim  $C_j$ . Therefore, the investment required in the risky asset (or the market exposure) at date  $t$  in order to achieve the payoffs  $C(S_H)$  at the horizon date will be

$$x_t(S_t) = \sum_{j=1}^{J+1} w_j x_{t,j}(S_t) \quad (7.27)$$

Thus, we can perform a simple linear least squares estimation for the  $w_j$ s subject to nonnegativity constraints (i.e.  $w_j \geq 0$ ).<sup>3</sup> Once the  $w_j$  have been estimated, we can easily obtain the market exposure for the nearest efficient strategy at each time point. Following earlier discussions, we then have  $x_t$  as the estimate of  $\phi_t^*$ .

## 7.4 Numerical Examples

In this section, we implement our technique on some simple portfolio strategies through Monte Carlo simulations. For the purpose of variance reduction, we have used stratified sampling technique (see Curran [31] and Moro [88]) in the simulation instead of the crude Monte Carlo method.<sup>4</sup>

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<sup>3</sup>A more detailed description on this constrained optimisation problem can be found in Hodges [59]. It is also suggested that the Singular Value Decomposition technique can provide a robust way of performing the computations.

<sup>4</sup>Stratified sampling is similar to quasi-random sampling but is easier to implement. If  $M$  is the number of simulation paths, stratified sampling has the advantage that the



The set of values used in the simulations are

riskless interest rate ,  $r = 0.06$

annualised drift rate of risky asset ,  $\mu = 0.12$

annualised standard deviation ,  $\sigma = 0.20$

The initial reference index is  $S_0 = 100$  and the initial wealth is also normalised as  $W_0 = 100$ . We consider an investment horizon of 5 years and assume that there are only 36 monthly observations (i.e. 3 years). In estimating the nearest efficient portfolio, we use  $J + 1 = 7$  options.

In most cases, particularly in hedge funds, the nature of the manager's strategy would not be known exactly to the investor or the analyst. Therefore, we could only look at the realised path and measure the performance as well as possible. Note that if we were in Dybvig's position of knowing in advance the nature of the strategy, we could apply our method to multiple simulations. In order to assess how our technique works, we need to know a strategy globally and work out pathwise inefficiency cost across all paths.

One well-known strategy is the stop-loss strategy. The investment rule under the stop-loss strategy is to invest the entire portfolio in the risky asset until the price of the risky asset reaches or falls below the preset level (or the floor). When that happens, the rule is to switch from 100% risky investment to 100% riskless investment.

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standard error is proportional to  $1/M$  rather than  $1/\sqrt{M}$  as it would be in the crude Monte Carlo simulation method.

Figures 7.1 and 7.2 plot the frequency polygon and histogram of the APEL of a 5-year stop-loss strategy. In this case, when the limit level is low, we expect that there will be more realisations that would be exposed as efficient. The spikes observed in the figures represent the chances of ‘not getting caught’. As the limit level moves higher, the spikes will diminish and eventually vanish.

When comparing the average pathwise cost with Dybvig’s actual cost, Figure 7.3 shows that at relatively lower limit levels, the average pathwise costs are lower than the actual ones. However, as the limit level increases, the ex-post inefficiency costs will tend to be overestimated. This should not be surprising because when the limit level is close to the initial wealth, the strategy becomes very sensitive to the reference index. As a result, the estimation procedure that tends to ‘smooth out’ the exposure will be more likely to create large departures from the values of the strategy.

We have also implemented on the lock-in strategy which we do not report in any detail here since it is simply the opposite of the stop-loss strategy.

## 7.5 Generalisations and Extensions

The results of our numerical experiments are quite encouraging. Although the number of observations of the portfolio’s composition is small (as it is usually the case in the hedge fund analysis), we are able to recover some interesting information about the managers strategy.

Our technique is based on observing what the manager actually did along the



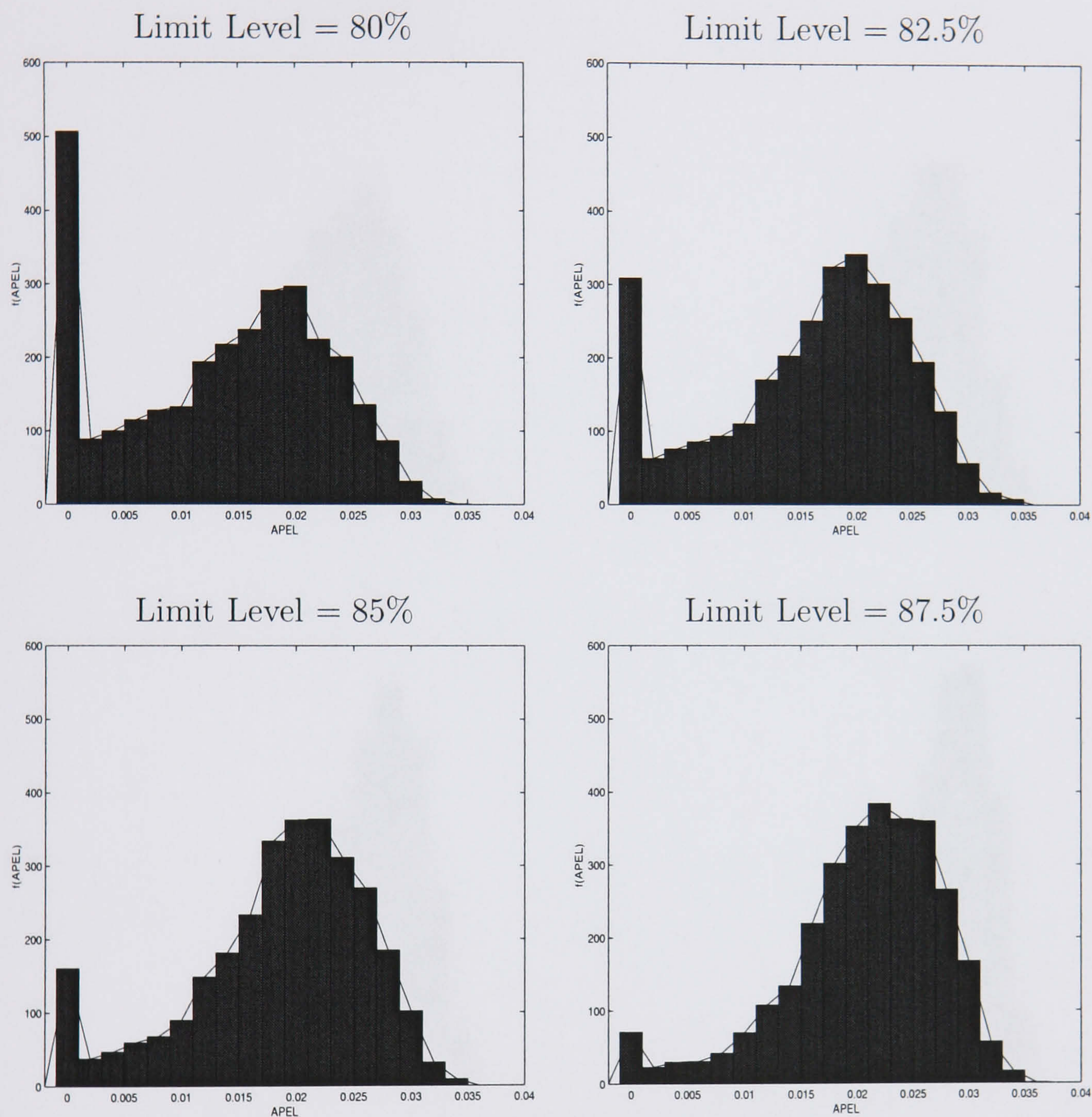


Figure 7.1: Frequency polygon and histogram of the APEL of a 5-year stop-loss strategy. The model parameter values are:  $\mu = 0.12$ ,  $r = 0.06$ ,  $\sigma = 0.2$ . The number of simulated paths for the risky asset is set equal to 3000. The initial reference index and the wealth are assumed equal, i.e.  $W_0 = S_0 = 100$ . The investment horizon is 5 years and there are total 36 (3 years) monthly observations.



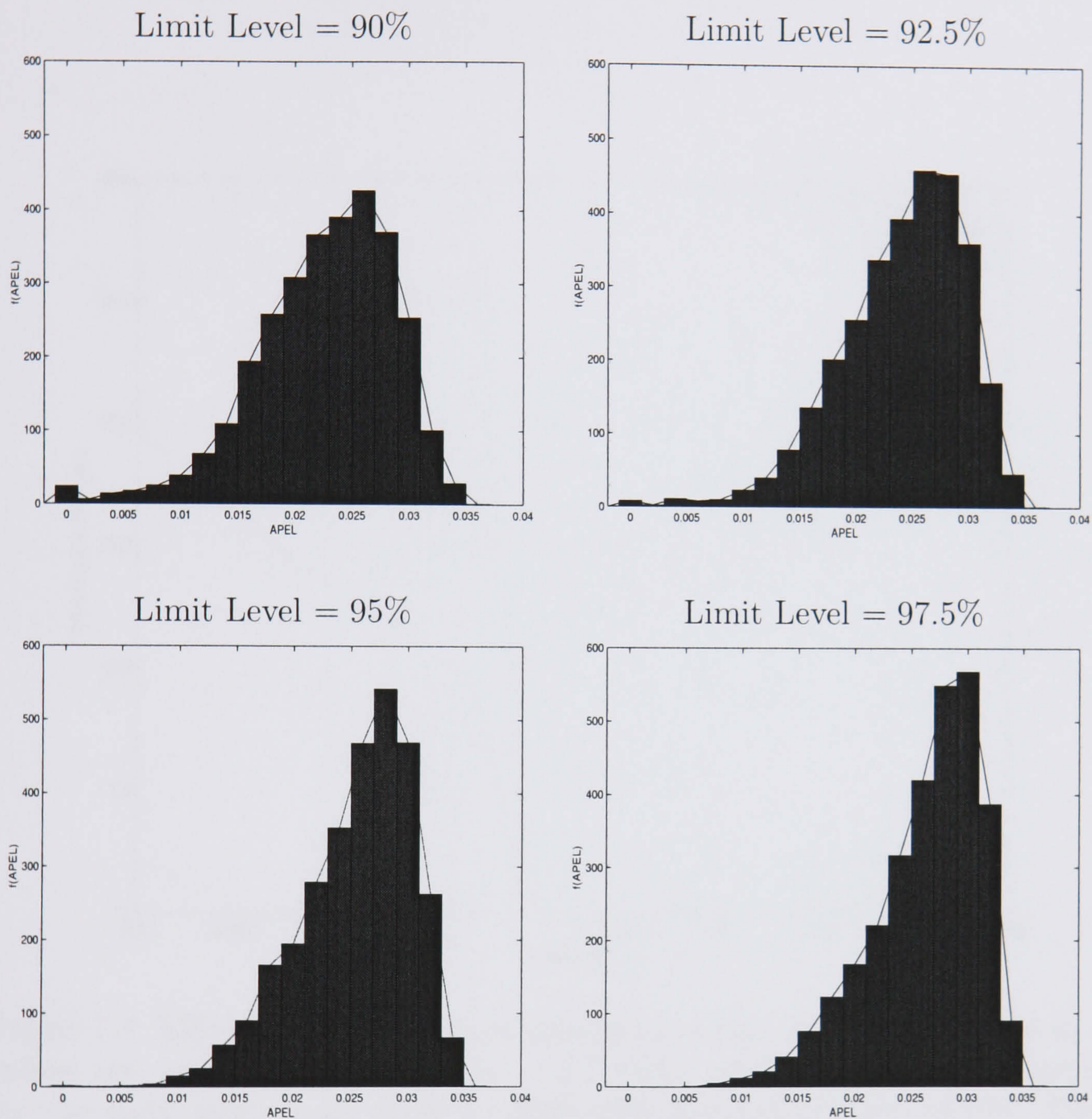


Figure 7.2: Frequency polygon and histogram of the APEL of a 5-year stop-loss strategy (continued). The model parameter values are:  $\mu = 0.12$ ,  $r = 0.06$ ,  $\sigma = 0.2$ . The number of simulated paths for the risky asset is set equal to 3000 with the initial value of 100. The investment horizon is 5 years and the number of monthly observations for the portfolio is set equal to 36 (3 years).



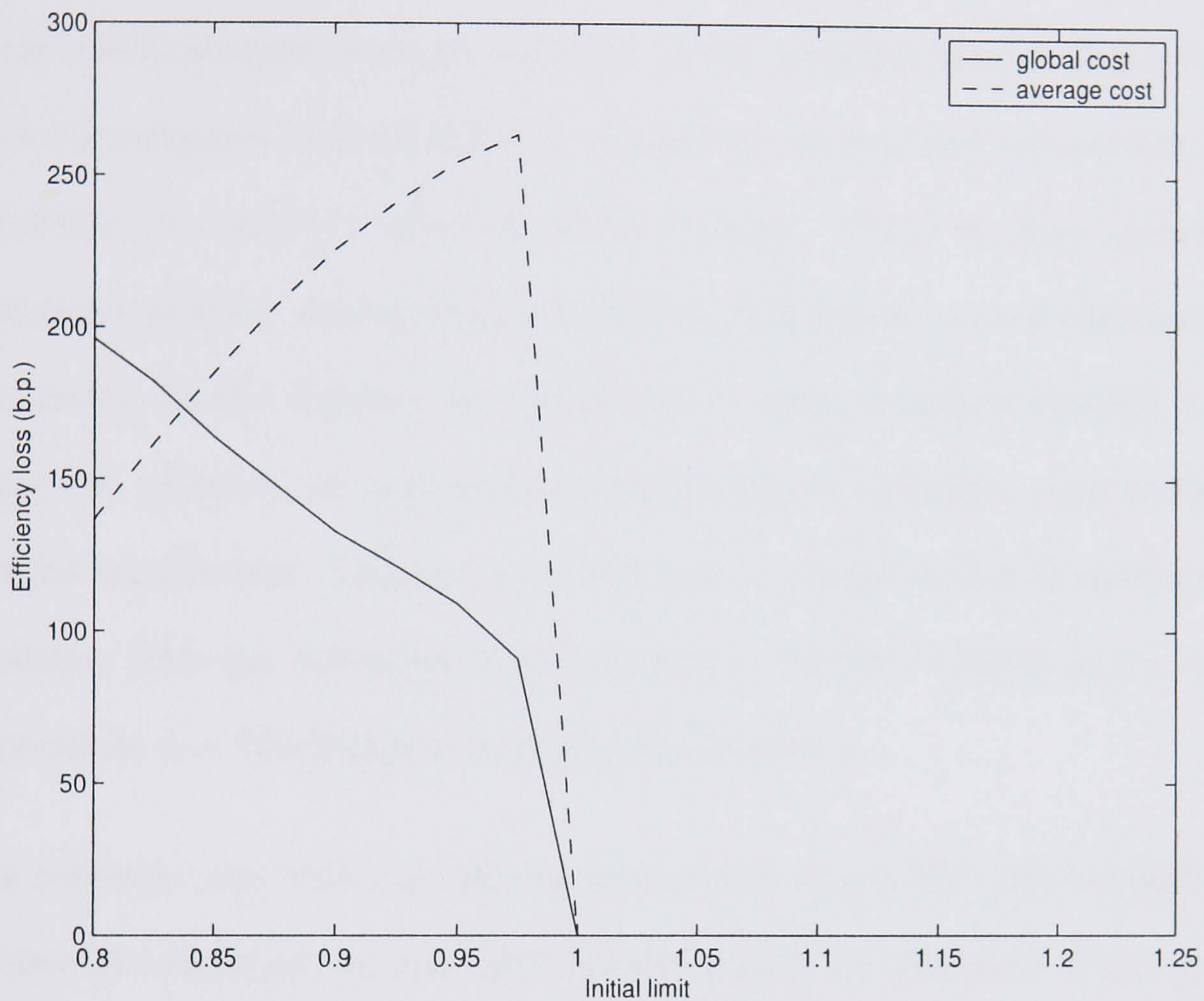


Figure 7.3: Efficiency loss of a 5-year stop-loss strategy. The model parameter values are:  $\mu = 0.12$ ,  $r = 0.06$ ,  $\sigma = 0.2$ . The number of simulated paths for the risky asset is set equal to 3000 with the initial value of 100. The investment horizon is 5 years and the number of monthly observations for the portfolio is set equal to 36 (3 years). The solid line indicates the actual costs (Dybvig's) and the dashed line indicates the average pathwise costs.



realised path. Thus, it can be greatly improved if we knew what stand the investor has taken in his/her preferences. Even the knowledge of the shape of the payoff function (e.g. concave or convex) should help estimate more precisely the nearest efficient strategy.

Our model assumes that  $\mu$  is constant. This assumption is critical in that it implies the reference index is efficient. Therefore, we can find the positive state-price density to correctly price the index options. Thus, we have ignored the inefficiencies possibly arising from model misspecification or poor representation of the proxy. In this respect, our model can be easily extended to allow a time-varying risk premium, as long as the price process is consistent with some form of market equilibrium. Otherwise we will have a market which is stochastically dominated. This also corresponds to our analysis on the evolution of the market risk premium (i.e. the Burgers' equation) in Chapter 5.

In principle, our work can also be generalised to models with multiple state variables. For example, we may allow interest rates, inflation and volatility to be stochastic. The state-price density framework can still work as long as the market is still assumed to be complete.

The analysis in this chapter ignored the important issue of forecasting ability. A manager with superior ability to time the market based on the forecast of future returns may justifiably lead the fund to depart from the benchmark strategies which we have labelled as efficient. In fact, these benchmark strategies will only be efficient in a world characterised by the kinds of equilibrium we have described in Chapter 5. When the manager has some degree of forecasting ability, our proposed



performance measure can still provide a neutral yardstick and can be interpreted as the opportunity cost of departing from what would be efficient without such ability.

The reasons why the manager may not follow the efficient strategies in practice could possibly be justified if we had an imperfect market setting. For example, as shown in Jouini and Kallal [69], when market frictions, such as different borrowing and lending rates due to asymmetries of information, short selling costs, and bid-ask spreads, are taken into account, a correct performance measure must trade off the additional frictional costs of alternative investment strategies against their incremental benefit from diversification. Thus an efficient strategy in frictionless markets might become inefficient as bid-ask spreads are introduced. On the other hand, high borrowing costs, especially if they increase with leverage, can rationalise some strategies such as the stop-loss rule that are inefficient in frictionless markets.

Finally, an important issue of transaction costs has also been ignored in our analysis. We might be able to take advantage on the recent advances on intertemporal portfolio choice and on the replication of options positions under transaction costs. (See for example, Davis and Norman [33] and Hodges and Neuberger [62].) A recent paper by Pelsser and Vorst [91] also demonstrates how Dybvig's model of portfolio efficiency will become invalid when transactions costs are introduced.

Our analysis provides an initial exploration of the problem in a complete market setting. Further extensions give a fruitful area for future research.

## 7.6 Chapter Summary and Conclusions

This chapter has proposed a new continuous-time performance measure based on an intertemporal analysis and also showed how a contingent claims estimation procedure can be employed for evaluating the opportunity costs implied by some inefficient dynamic portfolio strategies. When the nature of the strategy is unknown, we are able to recover the nearest efficient strategy providing a similar distribution of outcomes from a handful of data along the realised path. For a pre-defined dynamic strategy, our approach enables us to gain, at least numerically, some insight into how the strategy might have performed in the ex-post sense.

Formally, the analysis is based on fairly strong assumptions, but seems remarkably robust in applications where they are violated. We have hoped initially to be able to represent the actual cost as an average of pathwise costs. However, there is no such exact relationship corresponding to the rather natural way in which we have measured the pathwise cost. It remains unclear whether an identity of this link can be established with some suitable realisation of the pathwise cost.



# Chapter 8

## Conclusions and Future Directions

In this thesis, we have investigated some important issues relating to dynamic portfolio management from various viewpoints. Our modelling methods were based on the assumption that the market is complete and were conducted in a continuous-time intertemporal framework.

### 8.1 Summary and Conclusions

In Chapter 3, issues relating to time-invariant minimum guaranteed portfolio strategies were discussed and analysed in a simple economy. Given the definition and the properties of the strategies in question, a general form of the wealth function and the associated strategy were established in a Black-Scholes world in which both interest rate and volatility are constant and consumption only occurs at the horizon. The efficient strategies were then further identified and characterised based on the necessary conditions for efficiency, i.e. non-negativity, mono-

tonicity and path-independence. It was shown that, based on our assumptions, there are two types of time-invariant strategies which are efficient: (1) the constant proportion portfolio strategy; and (2) the level-dependent portfolio strategy. The constant policy extends the popularised constant proportion portfolio insurance (CPPI) program to include the interesting contrarian strategy. The level-dependent policy is of a contrarian type, and is efficient only when the interest rate is fairly low.

In Chapter 4, the dynamic asset allocation problem was studied in a more complicated setting where the interest rates were stochastic. Given a fairly general one-factor short rate process and the stock index dynamics, the equilibrium conditions, under which the coefficients of the processes must satisfy in a conventional representative agent economy, were established. The solutions of market parameters for some special cases were then further identified. These results were utilised to solve the non-representative agent's optimal portfolio choice problem. A special case where the short interest rate follows the Vasicek's [103] mean-reverting stochastic process was analysed. Two alternative utility functions were assumed for a non-representative agent: (1) a CRRA utility which is independent of the interest rate; and (2) a utility which is of the same form as the representative agent but with different risk aversion. The optimal portfolio choice problems for the two utility assumptions were solved explicitly by applying the dual approach. The results agreed with the conventional wisdom that the more risk averse the investor, the more the demand for the bond. The demand functions for the two cases behave quite differently, but will converge as the time to maturity increases.



In Chapter 5, the behaviour of the risk premium of the market portfolio was examined. The asset price dynamics were formulated in such a way that the market risk premium could vary over time in a Black-Scholes economy. The path-independence argument was used to help develop the theoretical constraints on the coefficients of the market portfolio. The utility function of the representative agent was formulated under two alternative assumptions: (1) a utility over a single consumption at the horizon date; and (2) a utility over the continuous consumption for a finite horizon. The equilibrium conditions were presented in terms of partial differential equations. These PDEs were then solved explicitly for the time-homogeneous case. It was shown that the presence of intermediate consumption could have a drastic effect on the time-varying behaviour of the risk premium. A decreasing form which corresponds to a mean reversion kind of behaviour could only exist when the intermediate consumption was introduced.

Chapters 6 and 7 investigated issues concerning portfolio efficiency and performance evaluation. In Chapter 6, the result developed in Chapter 5, that the Burgers' equation must be satisfied in a Black-Scholes single consumption economy, was used as the hindsight. Various functional forms were postulated in such a way that they were stationary but violated the Burgers' equation. Under these assumptions, the market portfolio would have become inefficient investment. The magnitude of the inefficiency was computed by implementing Dybvig's payoff distribution pricing model (PDPM). The Monte Carlo simulation technique was employed for the implementations. The experiments confirmed our prediction. However, the degree of inefficiency is insignificant, e.g. of the scale of 20 basis points or less, even for a 40-year horizon.

In Chapter 7, a new utility based performance measure was proposed. The idea was that the inefficiency cost of a portfolio strategy could be represented in terms of the expected total loss of certainty equivalent wealth over the investment horizon. The new measure was then derived from an intertemporal framework. In the context of ex-post performance evaluation, the continuous-time measure was approximated in a discrete-time manner and a contingent claim approach was utilised to estimate the nearest efficient portfolio strategy. Some simple dynamic strategies such as the stop-loss rule and the lock-in strategy were examined. Our simulations demonstrated that, in general, the average ex-post inefficiency cost is higher than the ex-ante cost computed from Dybvig's PDPM. However, when the initial limit level is at a relatively low level, there is a higher chance that the manager would be exposed as if they had behaved efficiently, and therefore the ex-ante cost would turn out to be higher than the average ex-post cost.

## 8.2 Limitations

Our analyses in this thesis were primarily conducted in a simple economy. Except in Chapter 4, we have assumed that the agent (or investor) maximises the expected utility only over the terminal wealth. This assumption could potentially have influenced the analysis and consequently the results. A more complex setting such as the one employed in Chapter 5 should result in richer behaviour in the optimal consumption and portfolio choice problem.

Moreover, it was assumed, except in Chapter 5, that interest rates were deterministic and volatility was constant. These assumptions greatly simplified the



modelling process and enabled us to derive stronger results. However, a vast body of empirical work clearly indicates the need for more complex models that relax these assumptions and consider more realistic situations such as multiple state variables, stochastic interest rates, stochastic volatility, jumps, inflation and so on.

Finally, our performance evaluation was based on Dybvig's payoff distribution pricing approach in a perfect market setting. As shown by Jouini and Kallal [69] and Pelsser and Vorst [91], the presence of market frictions such as short-selling costs, bid-ask spreads and transaction costs etc would significantly distort the results of Dybvig. As a consequence, the performance measure derived in a frictionless market may no longer apply.

## 8.3 Directions for Future Research

The work carried out in this thesis can be extended in several ways. The dynamic asset allocation problem under stochastic interest rates, as analysed in Chapter 4, was modelled in a single consumption economy. It would be worthwhile to generalise our approach to encompass the situation in which there is continuous consumption, such as the kind of equilibrium setting employed in Chapter 5.

The assumption that there are no transactions costs is apparently a very restrictive one since in reality transactions costs play a crucial role in the dynamic portfolio trading. Recent progress has been made on the intertemporal portfolio choice problems under transactions costs (see for example, Davis and Norman [33]) and on the optimal replication of contingent claims under transactions costs

(see for example, Hodges and Neuberger [62]). Therefore, it remains to be investigated what implications they provide for evaluating the performance of dynamic portfolio strategies.

Finally, the new performance measure proposed in Chapter 7 shared a similar concept as Dybvig's measure but was derived in a rather different way. It would be a significant contribution to demonstrate how the global inefficiency cost such as Dybvig's measure is related to the average of differently calculated pathwise costs, or whether there can be no strong relationship between the two at all.



# Appendix A

## The Brownian Bridge Processes

A Brownian bridge process (see for example, Kloeden, Platen and Schurz [74], p.59) is a modification of the Wiener process and has sample paths which all pass through the same initial point  $x$  at time  $t_a$  and a given point  $y$  at a later time  $t_b$ . This process  $B_{t_a, x}^{t_b, y}$  is defined sample pathwise for  $t_a \leq t \leq t_b$  by

$$B_{t_a, x}^{t_b, y}(t, w) = x + W(t, w) + \frac{t - t_a}{t_b - t_a}[y - x - W(t_b, w)] \quad (\text{A.1})$$

where  $W$  is a standard Wiener process. Equation (A.1) is sometimes called a tied-down Wiener process. It can be considered as a kind of conditional Wiener process and is determined uniquely by its mean and covariance, which are

$$\mathbb{E}[B_t] = x + \frac{t - t_a}{t_b - t_a}(y - x), \quad (\text{A.2})$$

$$\text{Cov}[B_s B_t] = \min\{\tau_1, \tau_2\} - \frac{\tau_1 \cdot \tau_2}{\tau} \quad (\text{A.3})$$

where  $\tau_1 = s - t_a$ ,  $\tau_2 = t - t_b$  and  $\tau = t_b - t_a$  for  $t_a \leq s$ ,  $t \leq t_b$ , respectively.

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